A Study Of Rodrigues' Formulae And Generating Relations Of Special Functions And Dual Series Equations

THESIS PRESENTED
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C E R T I F I C A T E

This is to certify that Arvind Kumar Agarwal has duly completed his thesis entitled "A STUDY OF RODRIGUES' FORMULAE AND GENERATING RELATIONS OF SPECIAL FUNCTIONS AND DUAL SERIES EQUATIONS" for the degree of Ph.D. of Bundelkhand University, Jhansi and his thesis is upto the mark both in its academic contents and quality of presentation.

I further certify that this work has originally been done by him under my supervision and guidance.

Dated : 30-5-1991

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DECLARATION

I hereby state that the present work "A STUDY OF RODRIGUES' FORMULAE AND GENERATING RELATIONS OF SPECIAL FUNCTIONS AND DUAL SERIES EQUATIONS" has been carried out by me under the supervision and guidance of Dr. P.N. Shrivastava, at the Department of Mathematics and Statistics, Bundelkhand University, Jhansi, and to the best of my knowledge, a similar work has not been done anywhere so far.

Dated : 30-5-1991

Agarwal

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P R E F A C E

The present thesis consists of seven Chapters, numbered I, II ...... VII. Each Chapter is divided into several sections (progressively numbered as 1.1, 1.2,...). The formulae and equations are numbered progressively within each section. For example (5.3.1) denotes the Ist formula or equation in 3rd section of the 5th Chapter.

When it comes to express the heart-felt gratitude towards those who were life and soul to this work, my situation is aptly summed-up by the lines 'when the heart is full, the tongue is silent' words if they could be adequately used, would perhaps still not suffice in bringingforth the totality of my gratefulness for all those who helped in building up this thesis to its present status.

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(ARVIND KUMAR AGARWAL)

CONTENTS

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CHAPTER I

A BRIEF HISTORICAL REVIEW

1.1 Introduction : In this Chapter we propose to give a brief historical account of some of the work done in the field of hypergeometric type of functions and polynomials. No attempt has been made to give a comprehensive review of the entire literature on the subject due to its vastness but only those aspects which are directly related to my work have been dealt with in some details.

Special functions have many physical and technical applications and a continuously growing importance since they are closely connected with the general theory of orthogonal polynomials and related problems of mechanical quadrature. Besides this, they have intrinsic mathematical interest.

So far, a number of mathematicians have studied various type of hypergeometric functions and polynomials viz., Laguerre, Hermite, Bessel, Jacobi, Hahn etc. and obtained several properties by using different techniques. Various generalizations of these functions and polynomials are also been introduced. Generally, these investigations have been made through their generating functions, Rodrigues' type of formulae, power series representations, operational representations, differential equations,

expansions, recurrence relations, difference equations and similar individual characteristic properties.

In the study of hypergeometric type of functions and polynomials different type of operations viz., differential, integral, difference, q-difference, shift operators and their various combinations also play a very important role. By using these operators several mathematicians, notably, Appell [12], Jackson [52], Toscano [131,132,135], Carlitz [23,25], R.P.Agrawal [8,9], Al-Salam [10], Shrivastava [94,95,96], Mittal [74], Rota, Kahaner and Odlyzko [86], H.C. Agrawal [5,6,7] and others derived numerous properties. We shall make frequent use of the difference, shift and q-difference operators in this thesis.

1.2 Ordinary and Basic Hypergeometric Functions of One Variable : In an attempt to generalized ordinary geometric series $\sum_{n=0}^{\infty} x^n$, several attempts were made before the nineteenth century and various similar series were introduced. But the proper form was developed by the famous German mathematician Carl Fridrich Gauss [46] in the year 1812, in his thesis presented at Gottingen as

$$(1.2.1) \qquad \sum_{n=0}^{\infty} \frac{(a)_n \ (b)_n}{(c)_n} \cdot \frac{z^n}{n!} = 1+ \frac{abz}{c1!} + \frac{a(a+1)\ b(b+1)\ z^2}{c(c+1)\ 2!} + \ldots,$$

where $(a)_n$ is Pochhammer symbol (also known as factorial function) given by

(1.2.2) $(a)_n = a(a+1) \ldots (a+n-1); \qquad n \geq 1,$

$(a)_0 = 1; \qquad a \neq 0.$

The above series (1.2.1) is called Gauss's series or ordinary hypergeometric series. It is represented by the symbol $_2F_1(a,b;c;z)$ the well known Gauss hypergeometric function.

A natural generalization of $_2F_1$ is the generalized hypergeometric function, the so-called $_AF_B$, which is defined in the following manner

(1.2.3) $$_AF_B((a);(b);z) = {}_AF_B\left[\begin{matrix}(a);\\(b);\end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{((a))_n}{((b))_n} \cdot \frac{z^n}{n!},$$

where for brevity $((a))_n$ stands for $(a_1)_n(a_2)_n \ldots (a_A)_n$, with similar representation for $((b))_n$. It is being assumed that no denominator parameter b_j is zero or a negative integer, as in that case the series is not defined.

The series on the right hand side of (1.2.3) is absolutely convergent for all values of z real or complex, when $A \leq B$. Also, when $A=B+1$, the series is convergent if $|z| < 1$. It converges when $z = 1$, if

$$\text{Re}\left[\sum_{j=1}^{B} b_j - \sum_{j=1}^{A} a_j\right] > 0$$

and when $z = -1$, if

$$\text{Re}\left[\sum_{j=1}^{B} b_j - \sum_{j=1}^{A} a_j\right] > -1.$$

If $A > B+1$, the series never converges except when $z = 0$, and the function is only defined when the series terminates.

If any of the numerator parameter a_j, in (1.2.3), is a negative integer, the series terminates and the function reduces to a polynomial.

A set of polynomials $\{P_n(x)\}$; $n=0,1,2,3,\ldots$, is called a simple set if $P_n(x)$ is of degree precisely n in x, so that the set contains one polynomial of each degree.

In our analysis we shall make the use of Laguerre, Jacobi and Hahn polynomials which are defined by

(1.2.4) $L_n^{(a)}(x) = \frac{(1+a)_n}{n!}\ {}_1F_1(-n; 1+a; x)$, [85, p.200]

(1.2.5) $P_n^{(a,b)}(x) = \frac{(1+a)_n}{n!}\ {}_2F_1\left[\begin{matrix} -n, 1+a+b+n; & \frac{1-x}{2} \\ 1+a; & \end{matrix}\right]$ [85,p.254]

and

(1.2.6) $Q_n(x; a, b, N) = {}_3F_2\left[\begin{matrix} -n, & 1+a+b+n, -x; & 1 \\ & 1+a, -N; & \end{matrix}\right]$ [119,1.9(5)]

In 1847, E.Heine [49] generalized the Gauss' function in a different direction. He defined a basic number as

(1.2.7) $(a;q) \equiv a_q = (1-q^a)/\ (1-q)$,

where q and a are real or complex numbers, so that as $q \to 1$, $a_q \to a$. Using this concept he defined the basic-analogue of the Gauss' function ${}_2F_1(a,b;c;z)$ as

$$(1.2.8) \quad {}_2\phi_1(a,b;c;q,q) = \sum_{n=0}^{\infty} \frac{(a;q)_n \, (b;q)_n}{(c;q)_n} \cdot \frac{z^n}{n!} \quad (q;q)_n$$

$$= 1 + \frac{(1-q^a)(1-q^b)}{(1-q^c)(1-q)} z + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q^c)(1-q^{c+1})(1-q)(1-q^2)} z^2 + \ldots,$$

where

$$(1.2.9) \quad (a;q)_n = (1-q^a)(1-q^{a+1})\ldots(1-q^{a+n-1})/(1-q)^n; \; n \geqslant 1,$$

$$(a;q)_o = 1 \; ; \; a \neq o,$$

$$(q;q)_n = (1-q)(1-q^2)\ldots(1-q^n) \; / \; (1-q)^n,$$

$$|q| < 1 \quad \text{and} \quad |z| < 1.$$

Clearly as $q \to 1$ the series (1.2.8) reduces to (1.2.1).

The general basic hypergeometric function analogous to (1.2.3) is defined as

$$(1.2.10) \quad {}_A\phi_B((a);(b);q,z) = \sum_{n=0}^{\infty} \frac{((a);q)_n}{((b);q)_n} \cdot \frac{z^n}{(q;q)_n}.$$

The series in (1.2.10) converges for all z when $|q| < 1$, $A+1 < B$ and no zeros appear in the denominator, and for $|z| < 1$ when $A+1 = B$. The case $|q| > 1$ can be transformed to $|q| < 1$ since

$$(a;q)_n = (-)^n a^n q^{n(n-1)/2} (a^{-1}; q^{-1})_n ,$$

when $|q| < 1$ and $A+1 > B$, the series diverges for all $z \neq 0$ unless terminates, which happens with any $a_j = q^{-k}$ and none of the b_j are of this form.

1.3. Hypergeometric Functions of Two Variables : The great success of the theory of hypergeometric functions of a single variable has motivated the development of a corresponding theory in two and more variables. Multiple hypergeometric functions has been found to be very useful in mathematical physics and statistical problems.

P. Appell [13] was the first author to treat this matter on a symmetric basis. In this year 1880, he defined the four functions given below which bear his name.

$$(1.3.1) \quad F_1(a;b,c;d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_m (c)_n}{(d)_{m+n}} \cdot \frac{x^m y^n}{m!n!} ;$$

$$(|x| < 1 , \quad |y| < 1) ,$$

$$(1.3.2) \quad F_2(a;b,c;d,e;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_m (c)_n}{(d)_m (e)_n} \cdot \frac{x^m y^n}{m!n!} ;$$

$$(|x| + |y| < 1) ,$$

$$(1.3.3)\quad F_3(a,b;c,d;e;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_n (c)_m (d)_n}{(e)_{m+n}} \cdot \frac{x^m y^n}{m!n!} \;;$$

$$(|x| \angle 1, \quad |y| \angle 1)$$

and

$$(1.3.4)\quad F_4(a,b:c,d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(c)_m (d)_n} \cdot \frac{x^m y^n}{m!n!} \;;$$

$$(\sqrt{|x|} + \sqrt{|y|} \angle 1).$$

The Appell functions posses various confluent forms which are analogues to the confluent hypergeometric functions in the case of one variable. P. Humbert [53] in 1920, first discussed seven such functions. We give here only those which have been used in subsequent chapters (for the rest see [40]) :

$$(1.3.5)\quad \phi_1 (a:b;c;x,y) = \lim_{u \to 0} F_1(a:b,1/u;c;x,uy)$$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_m}{(c)_{m+n}} \cdot \frac{x^m y^n}{m!\, n!} \;,$$

$$(1.3.6)\quad \Psi_2(a:b,c;x,y) = \lim_{u \to 0} F_2(a:1/u,1/u;b,c;\; ux,\; uy)$$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}}{(b)_m (c)_n} \cdot \frac{x^m y^n}{m!\, n!}$$

and

$$(1.3.7) \quad \Xi_1(a,b;c;d;x,y) = \lim_{u \to 0} F_3(a,b;c,1/u;d;x,uy)$$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_n (c)_m}{(d)_{m+n}} \cdot \frac{x^m y^n}{m!\, n!} \,.$$

In the course of a series of papers extending over the period of fifty years from 1889 to 1939, all the double hypergeometric functions of the second order were systematically investigated. J. Horn's final list [51] consisted of fourteen complete (non-confluent) series and their twenty distinct limiting forms which include the four Appell functions and the seven Humbert functions. A reference to the complete list of these functions may be made in the pioneer work of A. Erdelyi etc. [40, p.224-227].

In 1921, Kampe de Feriet [58] introduced the generalized Appell functions so as to provide the double hypergeometric functions of higher order. He studied the following function which is named after him.

$$(1.3.8) \qquad F\left[\begin{array}{ll|l} A & a_1,\ldots,a_A & \\ B & b_1,\ b_1',\ldots,b_B,\ b_B' & x,\ y \\ C & c_1,\ldots,c_C & \\ D & d_1,\ d_1',\ldots,d_D,\ d_D' & \end{array}\right]$$

$$\sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{A} (a_j)_{m+n} \prod_{j=1}^{B} (b_j)_m (b'_j)_n}{\prod_{j=1}^{C} (c_j)_{m+n} \prod_{j=1}^{D} (d_j)_m (d'_j)_n} \cdot \frac{x^m y^n}{m!\ n!} .$$

A more compact notation for the Kampe de Feriet function was first devised by Burchnall and Chaundy [22], and as such the function defined by (1.3.8) is now usually written as

$$(1.3.9) \qquad F\left[\begin{matrix}(a):(b);(c);\\(f):(g);(h);\end{matrix} x,y\right] = \sum_{m,n=0}^{\infty} \frac{((a))_{m+n}\ ((b))_m\ ((c))_n}{((f))_{m+n}\ ((g))_m\ ((h))_n} \cdot \frac{x^m\ y^n}{m!\ n!} .$$

The series (1.3.9) converges for all values of the variables x and y if $A+B < C+D+1$ and for $|x| < 1$, $|y| < 1$ if $A = B+1$.

In our analysis we shall also use the following basic-analogue of hypergeometric functions of two variables :

$$(1.3.10) \qquad \phi \left[\begin{matrix}(a) & : & (b); & (c); \\ (f) & : & (g); & (h);\end{matrix}\ q;\ x,y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{((a);q)_{m+n}\ ((b);q)_m\ ((c);q)_n}{((f);q)_{m+n}\ ((g);q)_m\ ((h);q)_n} \cdot \frac{x^m\ y^n}{(q;q)_m\ (q;q)_n}$$

and

$$(1.3.11) \qquad \phi^* \left[\begin{matrix}(a) & : & (b); & (c); \\ (f) & : & (g); & (h);\end{matrix}\ q;\ x,y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{((a);q)_{m+n} \, ((b);q)_m \, ((c);q)_n}{((e);q)_{m+n} \, ((g);q)_m \, ((h);q)_n} \cdot \frac{x^m \, y^n}{(q;q)_m \, (q;q)_n} \, q^{m(m-1)/2}$$

1.4 <u>The Lauricella Functions</u> - In the beginning of the 19th century, several authors, for example Green [48], Hermite [50] and Didon [37] studied certain specialised multiple hypergeometric functions. But a systematic approach was made by Lauricella [63] in 1893, who, beginning with the Appell functions, introduced following four important functions which bear his name,

(1.4.1) $F_A^{(n)}[a:b_1,\ldots,b_n\ ;\ c_1,\ldots,c_n\ ;\ x_1,\ldots,x_n]$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a_1)_{m_1+\ldots+m_n} (b_1)_{m_1} \ldots (b_n)_{m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n}} \cdot \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!}\ ;$$

$$|x_1| + \ldots + |x_n| < 1\ ,$$

(1.4.2) $F_B^{(n)}[a_1,\ldots,a_n\ ;\ b_1,\ldots,b_n\ ;\ c\ ;\ x_1,\ldots,x_n]$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a_1)_{m_1} \ldots (a_n)_{m_n} (b_1)_{m_1} \ldots (b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}} \cdot \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!}$$

$$\max\{|x_1|\ ,\ldots,\ |x_n|\} < 1\ ;$$

(1.4.3) $F_C^{(n)}\ [a:b;c_1,\ldots,c_n\ ;\ x_1,\ldots,x_n]$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} (b)_{m_1+\ldots+m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n}} \cdot \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!} \;;$$

$$\sqrt{|x_1|} + \ldots + \sqrt{|x_n|} < 1 \;;$$

(1.4.4) $F_D^{(n)} [a:b_1,\ldots,b_n \;;\; c \;;\; x_1,\ldots,x_n]$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} (b_1)_{m_1} \ldots (b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}} \cdot \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!}$$

$$\max \{ |x_1| ,\ldots, |x_n| \} < 1 .$$

In his work Lauricell also indicated the existence of other multiple hypergeometric functions. In our analysis we shall use the following confluent forms of the Lauricella functions introduced by Humbert [53] and Erdelyi [39].

(1.4.5) $\Psi_2^{(n)} [a:c_1,\ldots,c_n \;;\; x_1,\ldots,x_n]$

$$= \lim_{u \to 0} F_A^{(n)} [a:1/u,\ldots,1/u \;;\; c_1,\ldots,c_n \;;\; ux_1,\ldots,ux_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n}} \cdot \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!}$$

and

(1.4.6) $\Phi_2^{(n)}[b_1,\ldots,b_n ; c ; x_1,\ldots,x_n]$

$$= \lim_{u \to 0} F_D^{(n)} [1/u : b_1,\ldots,b_n ; c ; ux_1,\ldots,ux_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(b_1)_{m_1} \ldots (b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}} \cdot \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!} ,$$

respectively.

1.5 <u>Difference, Shift and q-Difference Operators</u> : The most important conception of mathematical analysis is that of the function. In analysis we usually encounter two types of functions, first type of functions are those in which the variable x takes all possible values in a given interval. These functions belong to the domain of Infinitesimal Calculus. The second class of functions in which the variable assume only the discrete set of given value $x_0, x_1, \ldots, x_n$, are dealt with the methods of Finite Differences, although it may be applied to both the classes.

The origin of this calculus may be ascribed to Brook Taylor's <u>Methodus Incrementorum</u> (London, 1717), but the real founder of the theory was Jacobi Stirling who in his <u>Methodus Differentialis</u> (London, 1730) introduced the famous Stirling numbers.

Denoting the first difference of $f(u)$ by $\Delta_{u,h}$, we have

(1.5.1) $\Delta_{u,h} f(u) = f(u+h) - f(u)$,

where h is called the interval of differencing or increment.

The nth iterated difference of f(u) is given by

$$(1.5.2) \quad \Delta^{n}_{u,h}\,[f(u)] = \Delta_{u,h}\,[\,\Delta^{n-1}_{u,h}\; f(u)\,] \;;\; n = 1,2,\ldots .$$

An associated operator is the shift operator. A shift operator, written as $E_{u,h}$, is an operator which translates the argument (or parameter) of a polynomial by h. Hence

$$(1.5.3) \quad E^{a}_{u,h}\,[f(u)] = f(u + ah).$$

(1.5.1) and (1.5.3), yields the following relation between $\Delta_{u,h}$ and $E_{u,h}$

$$(1.5.4) \quad \Delta_{u,h} \equiv E_{u,h} - 1$$

which in turn gives us

$$(1.5.5) \quad \Delta^{n}_{u,h}\,[f(u)] = \sum_{m=0}^{n} \binom{n}{m} (-)^{n-m} f(u+mh) \;;\; n = 1,2,\ldots .$$

We also have

$$(1.5.6) \quad \Delta^{n}_{u,h}[f(u)g(u)] = \sum_{r=0}^{n} \binom{n}{r} \Delta^{n-r}_{u,h}\, f(u+rh)\, \Delta^{r}_{u,h}\, g(u) \;;$$

$$n = 1,2,\ldots .$$

Throughout the present work, we denote $\Delta_{u,1}$ and $E_{u,1}$ simply by Δ_u and E_u respectively. Thus, we have

$$(1.5.7)\quad \Delta_u\left[\,f(u)\,\right] = f\,(u+1) - f(u)$$

and

$$(1.5.8)\quad E_u^a\left[\,f(u)\,\right] = f(u+a).$$

There is another operator ∇_u which is closely related to Δ_u and is known as the backward difference operator. It is defined by

$$(1.5.9)\quad \nabla_u\left[\,f(u)\,\right] = f(u) - f(u-1).$$

In our present work we also make the use of the q-difference operator $(q^u\Delta_u)$ introduced earlier by Gould [47, (3.1)] (for h = 1),

$$(1.5.10)\quad (q^u\Delta_u)^n f(u) = q^{nu}\sum_{r=0}^{n}(-)^{n-r}\left[{n \atop r}\right]q^{r(r-1)/2}\, f(u+r),$$

and Leibnitz analogue is

$$(1.5.11)\quad (q^u\Delta_u)^n\left[f(u)g(u)\right] = \sum_{r=0}^{n}\left[{n \atop r}\right](q^u\Delta_u)^{n-r}\, f(u+r)$$

$$\times\ (q^u\Delta_u)^r\, g(u),$$

where

$$\left[{n \atop r}\right] = \frac{(q;q)_n}{(q;q)_r\,(q;q)_{n-r}} = \frac{(-)^r\,(q^{-n};\,q)_r}{(q;q)_r}\, q^{-r(r-1-2n)/2}.$$

1.6 Rodrigues' Formula and Its Generalizations : The Rodrigues type formulae have been widely used by numerous researchers in the past. The classical orthogonal polynomials have a generalized Rodrigues' formula of the form

$$(1.6.1) \quad F_n(x) = \frac{1}{k_n w(x)} D^n \left[w(x) X^n \right] ; \; n = 0, 1, 2, \ldots,$$

where $D \equiv \frac{d}{dx}$, k_n is a constant, X is a polynomial in x whose coefficient are independent of n, w(x) is the weight function and $F_n(x)$ is a polynomial of degree n in x.

Conversely, any system of orthogonal polynomials which satisfies (1.6.1) can be reduced to a classical set.

The Legendre, Laguerre and Hermite polynomials which satisfy (1.6.1) are the particular cases of the Rodrigues' formula. They are as follows :

$$(1.6.2) \quad P_n(x) = \frac{1}{2^n \, n!} D^n (x^2-1)^n ,$$

$$(1.6.3) \quad L_n^{(a)}(x) = \frac{1}{n!} x^{-a} e^x D^n (x^2-1)^n$$

and

$$(1.6.4) \quad H_n(x) = (-)^n e^{x^2} D^n (e^{-x^2}).$$

The classical orthogonal polynomials have also generalized Rodrigues type of formula of the form

$$(1.6.5)\quad f_n(x) = \frac{1}{k_n\, J(x)} \Delta_x^n \left[J(x-n)\ X(x)\ X(x-1) \ldots X(x-n+1) \right]; \quad n = 0,1,2,\ldots,$$

where k_n is a constant, $X(x)$ a polynomial in x whose coefficients are independent of n, $J(x)$ is any function and Δ_x is the operator of finite difference defined by (1.5.7).

Numerous polynomials and functions have been defined by the formulae analogous to Rodrigues' type of formula in special functions of Mathematical Physics. While defining various polynomials and functions, a number of researchers, notably, Appell [14], Burchnall [21], Chak [32], Al-Salam [10], Carlitz [25], Chaterjea [34], P.N. Shrivastava [94, 95, 97], Mittal [73], Patil and Thakare [80, 81, 82], Joshi and Prajapat [56, 57] etc. have used various combinations of differential operator viz., xD, x^kD, $ax^k + x^{k+1}D$ etc. The most common polynomials, functions and numbers defined by above operators are Truesdell type polynomials, Stirling numbers and generalized polynomials. But not much corresponding work has been done in terms of difference operators.

In 1949 Toscano [131], introduced the following Rodrigues' type of representation for generalized hypergeometric polynomial of one variable.

$$(1.6.6) \quad {}_{A+1}F_{B+1}\left[\begin{matrix} -n, (a)+u ; \\ u, (b)+u ; \end{matrix} \; t \right]$$

$$= (-)^n \frac{\Gamma(u) \;\; \Gamma((b)+u)\, t^{-u}}{\Gamma((a)+u)} \Delta_u^n \left[\frac{\Gamma((a)+u) \;\; t^u}{\Gamma(u) \;\; \Gamma((b)+u)} \right],$$

where for brevity $\Gamma((a)+u)$ stands for $\Gamma(a_1+u)\ldots\Gamma(a_A+u)$, with similar representation for $\Gamma((b)+u)$.

In similar manner Agrawal [6, 7] obtained

$$(1.6.7) \quad F^{(2)}\left[\begin{matrix} (a)+u+v : -m, & (b)+u ; -n, & (c)+v ; \\ (f)+u+v : & (g)+u ; & (h)+v ; \end{matrix} \; x, \; y \right]$$

$$= (-)^{m+n} \frac{\Gamma((f)+u+v) \;\; \Gamma((g)+u) \;\; \Gamma((h)+v)}{\Gamma((a)+u+v) \;\; \Gamma((b)+u) \;\; \Gamma((c)+v)} x^{-u} \; y^{-v}$$

$$\times \Delta_u^m \Delta_v^n \left[\frac{\Gamma((a)+u+v) \;\; \Gamma((b)+u) \;\; \Gamma((c)+v)}{\Gamma((f)+u+v) \;\; \Gamma((g)+u) \;\; \Gamma((h)+v)} \; x^u \; y^v \right],$$

and also gave a Rodrigues' type of formula for hypergeometric polynomials of three variables in terms of difference operators.

Toscano in the same paper [131] considered the following Rodrigues' type of formula

$$(1.6.8) \quad L_n^{(a)}(x) = (-)^n \frac{\Gamma(a+n+1)}{n!} x^{-a} \Delta_a^n \left[\frac{x^a}{\Gamma(a+1)} \right]$$

and

$$(1.6.9)\quad P_n^{(a,b)}(x) = (-)^n \frac{\Gamma(a+n+1)}{n!\,\Gamma(a+b+n+1)} \left(\frac{1-x}{2}\right)^{-a-1}$$

$$\times \Delta_a^n \left[\frac{\Gamma(a+b+n+1)}{\Gamma(a+1)} \left(\frac{1-x}{2}\right)^{a+1} \right],$$

for Laguerre and Jacobi polynomials respectively.

Later on Agrawal [1] also gave several properties of Laguerre polynomials by using the Rodrigues' type of formula (1.6.8).

In view of the above formulae (1.6.8) and (1.6.9) Toscano [133, 134] defined the generalized Laguerre and Jacobi polynomials as

$$(1.6.10)\quad L_n^{(a;v)}(x) = (-)^n \frac{\Gamma(a+nv+1)}{n!}\; x^{-a} \Delta_{a,v}^n \left[\frac{x^a}{\Gamma(a+1)} \right]$$

and

$$(1.6.11)\quad P_n^{(a,b;v)}(x) = (-)^n \frac{\Gamma(a+nv+1)}{n!\,\Gamma(a+b+n+1)} \left(\frac{1-x}{2}\right)^{-a-1}$$

$$\times \Delta_{a,v}^n \left[\frac{\Gamma(a+b+n+1)}{\Gamma(a+1)} \left(\frac{1-x}{2}\right)^{a+1} \right],$$

and also obtained several interesting results.

Following Toscano, other authors, notably, Soni, B.M. Agarwal, H.C. Agrawal, Karlin, McGregor etc. considered the Rodrigues type of formulae for several other well known polynomials and consequently derived their numerous properties :

$$(1.6.12) \quad Y_n(x;a,b) = \frac{(-)^n (-x/b)^{-a}}{\Gamma(a+n-1)} \Delta_a^n \left[\Gamma(a+n-1) \; (-x/b)^a \right], \qquad [106]$$

$$(1.6.13) \quad A_n^a(x) = \frac{(-)^n x^{n+a}}{n! \, \Gamma(1+a)} \Delta_a^n \left[x^{-a} \, \Gamma(1+a) \right], \qquad [3, (2.1)]$$

$$(1.6.14) \quad M_n^k(x;a,b) = \frac{(-)^{-n-a} (x/b)^{-a}}{\Gamma(kn+a-k-n+1)} \Delta_a^n \left[(-x/b)^a \right.$$

$$\left. \times \; \Gamma(kn+a-k-n+1) \right], \qquad [3, (7.1)]$$

$$(1.6.15) \quad l_n^a(x) = \frac{(-x)^{n+a}}{n! \, \Gamma(a-x)} \Delta_a^n \left[(-x)^{-a} \, \Gamma(a-x) \right], \qquad [5, (2.1)]$$

$$(1.6.16) \quad \binom{N}{n} \binom{x+a}{x} \binom{N-x+b}{N-x} Q_n(x;a,b,N)$$

$$= \binom{n+b}{n} \Delta_x^n \left[\binom{x+a}{a+n} \binom{N-x+b+n}{b+n} \right], \qquad [59, (1.8)]$$

$$(1.6.17) \quad m_n(x;b,c) = \frac{x!}{(b)_x} c^{-x-n} \Delta_x^n \left[c^x (b)_x / (x-n)! \right],$$

$$(1.6.18) \quad p_n(x;b,c,d) = \frac{x! \, (d)_x}{n! \, (b)_x \, (c)_x} \Delta_x^n \left[\frac{(b)_x \, (c)_x}{(x-n)! \, (d)_{x-n}} \right],$$

and

$$(1.6.19) \quad c_n(x;a) = \frac{x!}{a^x} \Delta_x^n \left[\frac{a^{x-n}}{(x-n)!} \right],$$

(for Rodrigues' formulae (1.6.17), (1.6.18) and (1.6.19) see [41])

where $y_n(x;a,b)$, $A_n^a(x)$, $M_n^k(x;a,b)$, $l_n^a(x)$, $Q_n(x;a,b,N)$, $m_n(x;b,c)$, $P_n(x;b,c,d)$ and $c_n(x;a)$ are known as Bessel, Associated Laguerre [112], generalized Bessel [33], generalized Tricomi [24], Hahn, Meixner, Tchebichef's, and Charlier's polynomials respectively.

1.7 Generating Functions : The term "Generating functions" was first introduced by P.S. Laplace [62] in 1812. Generating functions have great importance in the study of polynomial sets. For example, we shall define the Legendre polynomials $P_n(x)$ by

$$(1.7.1) \quad (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n ,$$

and the Hermite polynomials $H_n(x)$ by

$$(1.7.2) \quad \exp\ (2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)\ t^n}{n!}$$

The Laguerre polynomials $L_n^{(a)}(x)$ possess the generating relation

$$(1.7.3) \quad e^t\ {}_0F_1\ (-;1 + a\ ;\ -xt) = \sum_{n=0}^{\infty} \frac{t^n}{(1+a)_n} L_n^{(a)}(x) .$$

Generating functions are powerful tools in the investigations of the systems of polynomials sets and may be

used to determine differential, difference or pure recurrence relations and to evaluate certain integrals etc. It also helps in unifying treatment of polynomials. This fact is evinced by the works of Sheffer [92], Brenke [20], Rainville [84], Huft [52], Truesdell [138], Palas [76], Boas and Buck [18], Zeitlin [140] and Mittal [71, 72, 74].

Linear Generating Functions : If a function of two variables $G(x,t)$ has a formal power series (not necessarily convergent for $t \neq 0$) expansion in t of the form

$$(1.7.4) \qquad G(x,t) = \sum_{n=0}^{\infty} A_n f_n(x) t^n ,$$

where A_n ; $n = 0,1,2,\ldots,$ be a specified sequence independent of x and t, then we say that $G(x,t)$ is linear generating function or simply generating function of $f_n(x)$ and $G(x,t)$ is said to have generated the set $f_n(x)$. The above equation (1.7.1), (1.7.2) and (1.7.3) are the examples of linear generating functions.

Bilinear Generating Functions : Consider a three variable function $F(x,y,t)$ which possess a formal power series expansion in t of the form

$$(1.7.5) \qquad F(x,y,t) = \sum_{n=0}^{\infty} B_n f_n(x) f_n(y) t^n ,$$

where B_n ; $n = 0,1,2,\ldots,$ be a specified sequence independent

of x, y and t, then $F(x,y,t)$ is called a bilinear generating function for the set $f_n(x)$.

The more general form of bilinear generating function can be given by

$$(1.7.6) \quad F(x,y,t) = \sum_{n=0}^{\infty} B_n \, f_{a(n)}(x) \, f_{b(n)}(x) \, t^n \, , \qquad \text{what's f?}$$

where $a(n)$ and $b(n)$ are functions of n which are not necessarily equal.

Bilateral Generating Functions : If $G(x,y,t)$ a three variable function has a formal power series expansion in t of the form

$$(1.7.7) \quad G(x,y,t) = \sum_{n=0}^{\infty} D_n \, f_n(x) \, g_n(y) \, t^n \, ,$$

where D_n ; $n = 0,1,2,\ldots$, be a specified sequence independent of x, y and t, and $f_n(x)$ and $g_n(y)$ are the different types of functions. Then $G(x,y,t)$ is called a bilateral generating function of $f_n(x)$ and $g_n(x)$.

More generally, if $H(x,y,t)$ can be expanded in power series of t such that

$$(1.7.8) \quad H(x,y,t) = \sum_{n=0}^{\infty} D_n \, f_{a(n)}(x) \, g_{a(n)}(y) \, t^n \, ,$$

where $a(n)$ and $b(n)$ are functions of n which are not necessarily equal, we shall still call $H(x,y,t)$ is bilateral generating function for the functions $f_n(x)$ and $g_n(x)$.

Multivariable Generating Functions : In each of the above definitions, the sets generated are functions of only one variable. Now for the sets of the functions $f_n(x_1,x_2,\dots,x_r)$ and $g_n(y_1,y_2,\dots,y_s)$ of several variables, it is not difficult to extend the definitions of linear, bilinear and bilateral generating functions to include such multivariable generating functions as

$$(1.7.9)\quad F(x_1,\cdots,x_r\ ;\ t) = \sum_{n=0}^{\infty} A_n\ f_n(x_1,\cdots,x_r)t^n ,$$

$$(1.7.10)\quad F(x_1,\cdots,x_r\ ;\ y_1,\cdots,y_r\ ;\ t\ )$$

$$= \sum_{n=0}^{\infty} B_n\ f_{a(n)}(x_1,\cdots,x_r)\ f_{b(n)}(y_1,\cdots,y_r)\ t^n$$

and

$$(1.7.11)\quad H(x_1,\cdots,x_r\ ;\ y_1,\cdots,y_s\ ;\ t\ )$$

$$= \sum_{n=0}^{\infty} D_n\ f_{a(n)}(x_1,\cdots,x_r)\ g_{b(n)}(y_1,\cdots,y_s)\ t^n ,$$

respectively.

If

(1.7.12) $f_n(x_1, \cdots, x_r) = f_{a_1(n)}(x_1) \ldots f_{a_r(n)}(x_r)$,

(where $a_1(n), \cdots, a_r(n)$ are functions of n which are not necessarily equal) then multivariable generating function $G(x_1, \cdots, x_r ; t)$ given by (1.7.9), is said to be a multilinear generating function.

Further, if the functions occurring on the R.H.S. of (1.7.12) are all different, then multivariable generating function $G(x_1, \cdots, x_r ; t)$ given by (1.7.9) will be called a multilateral generating function.

Multiple Generating Functions : A natural further extension of the multivariable generating function (1.7.9) is a multiple generating function which may be defined formally by

(1.7.13) $F(x_1, \cdots, x_r ; t_1, \cdots, t_s)$

$$= \sum_{n_1, \cdots, n_s=0}^{\infty} A(n_1, \cdots, n_s)\, f_{n_1, \cdots, n_s}(x_1, \cdots, x_r) t_1^{n_1} \ldots, t_s^{n_s} ,$$

where the multiple sequence $A(n_1, \cdots, n_s)$ is independent of the variables $x_1, \cdots, x_r$ and $t_1, \cdots, t_s$.

Brief Survey : In the present thesis an attempt has been made to show that how effectively the difference, shift and the q-difference operators can be used in the hypergeometric type of functions and polynomials for finding Rodrigues' type of formulae, generating functions and for solving the problems of dual series equations in discrete variables.

In Chapter II we have introduced a generalized class of polynomials in the form of Rodrigues' type formula with the help of difference operator $\Delta_{u,h}$. For this generalized class of polynomials certain linear, bilinear and bilateral generating functions, operational formulae, Recurrence relations, finite expansions and other results have been obtained.

In Chapter III we have presented a Rodrigues' type formula in terms of the operator Δ_u for hypergeometric functions of two variables and subsequently have obtained a number of summation formulas and transformations for hypergeometric functions of two variables.

In Chapter IV, in view of the Rodrigues' type formula (1.6.6), a method has been developed by which several well known theorems viz., Gauss, Saalschütz, Dixon etc. can be proved easily. We have also considered a Rodrigues' type representations for basic-hypergeometric series of one variable. These representations have useful in obtaining

some transformations, summations, generating functions and three term relations involving basic hypergeometric series.

In Chapter V we have given a q-analogue of the Rodriques' type formula (1.6.7) and have used it to derive some transformations, summations formulas, generating functions and expansions involving basic hypergeometric series of two variables.

In Chapter VI we have established some general type of bilateral generating functions involving Laguerre/Jacobi polynomials with other functions of several variables. We have also derived certain multiple generating functions for the product of Laguerre/Jacobi polynomials and Lauricella functions by using the mathematical induction methods.

In Chapter VII we have considered a pair dual series equations involving Hahn polynomials in discrete variables as kernels. Some special cases have also been derived, which include some of the well known orthogonal polynomial in continuous variables as kernels.

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CHAPTER - II

ON GENERALIZED RODRIGUES' TYPE OF FORMULA.

2.1 Introduction : Rodrigues' type of formula have been the starting point of numerous researches in the past. In the beginning only the differential operator $\frac{d}{dx}$ was used in Rodrigues' formulae, but later on other operator viz., $x\frac{d}{dx}$, $x^k\frac{d}{dx}$, $x^k(a+x\frac{d}{dx})$ etc. were also used. As far as the use of differential operators in case of orthogonal polynomials is concerned perhaps for the first time they were used in 1934 for Laguerre polynomials. The Rodrigues' type of formula for Laguerre polynomials is given by

$$(2.1.1) \quad L_n^{(a)}(x) = \frac{(-)^n\,\Gamma(a+n+1)}{n!\,x^a}\,\Delta_a^n\left[\frac{x^a}{\Gamma(a+1)}\right].$$

Later on such type of representation were given for several other polynomials also for this one is advised to consult [3,41], L. Tascano [131, 132, 133, 134, 135, 136], B.M. Agrawal [1], S.L. Soni [106], G. Gasper [45], H.C. Agrawal [5] and B.P. Parashar [78, 79] are the main researchers who worked in this direction. They gave the Rodrigues' type formulae for different systems of polynomials in terms of difference operators and consequently find out

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several interesting known as well as new results. It is necessary to point out here that form such type of Rodrigues' formulae several important properties such as generating functions, recurrence relations, expansions etc. can be obtained very easily.

Tascano [133] also introduced and studied some new class of polynomials by generalizing Rodrigues' type formulae. For example, he gave the following generalized form of the system of polynomials $\{L_n^{(a)}(x)\}$ defined above by (2.1.1)

$$L_n^{(a;h)}(x) = \frac{(-)^n x^{-a} \Gamma(a+nh+1)}{n!} \Delta_{a,h}^{n} \left[\frac{x^a}{\Gamma(a+1)} \right].$$

In this Chapter our aim is to give a unified treatment to the subject. For that we study a new class of polynomials $\{G_n^{(u;h)}(x: (a);(b);(c):(d);(e);(f)) / n=0,1,2,\ldots\}$ defined by

$$(2.1.2) \quad G_n^{(u;h)}(x : (a);(b);(c):(d);(e);(f))$$

$$= (-)^n \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))} x^{-u} \Delta_{u,h} \left[\frac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))} x^u \right],$$

where $\Delta_{u,h} f(u) \equiv \Delta_{u,h} f(u) = f(u+h) - f(u)$

$$\Gamma((a)u+(b)n+(c)) = \Gamma(a_1u+b_1n+c_1) \ldots \Gamma(a_Au+b_An+c_A)$$

and

$$\Gamma((d)u+(f)n+(e)) = \Gamma(d_1u+f_1n+e_1)\ldots\Gamma(d_Du+f_Dn+e_D).$$

The motive behind introducing such a polynomial system is that it provides the unified study of polynomials. It includes as special cases not only a number of well known polynomials (a few of them mentioned below as illustration) but also provides their extension. Our aim is also to show that by using the difference operators, the results can be obtained very easily.

(2.1.3) $G_n^{(a;h)}(x:0;0;1 : 1,1,h)$

$$= (-)^n x^{-a}\Gamma(a+nh+1) \underset{a,h}{\Delta^n} \left[\frac{x^a}{\Gamma(a+1)} \right] = n!\, L_n^{(a;h)}(x) ,$$

(2.1.4) $\lim_{u\to 0} \left[G_p^{(u;1)}(x^{2k} : 0;0;1 : 1,1-1/2k;0) \right]$

$$= \frac{(-)^n\, p!}{(p+1)_p} H_{2pk}(x;k) ,$$

(2.1.5) $G_n^{(a;1)}(1/x : 1;0;1: 0;1;0).$

$$= \frac{(-)^n x^a}{\Gamma(a+1)} \underset{a,1}{\Delta} \left[x^{-a}\Gamma(a+1)\right] = n!\, x^{-n} A_n^{(a)}(x) ,$$

(2.1.6) $G_n^{(a;1)}(-1/x : 1;0;-x : 0;1;0)$

$$= \frac{(-)^{n+a} x^a}{\Gamma(a-x)} \underset{a,1}{\Delta^n} [\Gamma(a-x)(-x)^{-a}] = n!\ x^{-n}\ l_n^{(a)}(x) ,$$

$$(2.1.7) \quad G_n^{(a;h)}(\tfrac{1-x}{2} : 1;1;b+1:1;1;h) = \frac{(-)^n \Gamma(a+nh+1)}{\Gamma(a+b+n+1)} (\tfrac{1-x}{2})^{-a-1}$$

$$\times \underset{a,h}{\Delta^n} [\frac{\Gamma(a+b+n+1)}{\Gamma(a+1)} (\tfrac{1-x}{2})^{a+1}] = n!\ P_n^{(a,b;h)}(x)$$

$$(2.1.8) \quad G_n^{(a;1)}(-x/b:1;k-1;1-k:0;1;0) = \frac{(-)^n (-x/b)^{-a}}{\Gamma(kn+a-k-n+1)}$$

$$\times \underset{a,1}{\Delta^n} [(-x/b)^a \Gamma(kn+a-k-n+1)] = M_n^k(x;a;b) ,$$

$$(2.1.9) \quad \lim_{u\to 0} [G_n^{(u;1)}(x:1,1;1,0;1+a+b,p:1,1;1+a,q;0,0)]$$

$$= \frac{n!}{(1+a)_n} H_n^{(a,b)}(p,q;x) ,$$

$$(2.1.10) \quad \lim_{u\to 0} [G_n^{(u;1)}(1:1,1;1,0;1+a+b,-x:1,1;1+a,-N;0,0)]$$

$$= Q_n(x;a,b,N) ,$$

$$(2.1.11) \quad \lim_{u\to 0} [G_n^{(u;1)}(x:1,\ldots,1;0,\ldots,0;a_1,\ldots,a_p : 1,\ldots,1 ;$$

$$v+n, b_1,\ldots,b_q ; 1,0,\ldots,0] = n!\ S_n^{(v)}(x)$$

and

$$(2.1.12) \quad G_n^{(a;k)}\left(\frac{1-x}{2} : 1;1;1+b : 1;1;k\right) = n!\, J_n^{(a,b)}(x;k).$$

The polynomial $L_n^{(a;h)}(x)$ is introduced by Tascano [133] and is the generalization of Laguerre polynomial [85]. The same is later on reintroduced by Kaunhausar [61] also. He represented it by $Z_n^{(a)}(x;k)$. Later on Prabhakar and Suman [83] studied the polynomial $L_n^{(a,b)}(x)$ which is similar to $L_n^{(a;h)}(x)$ with a slight difference (in place of x they have taken $x^{1/h}$) only. Recently, Parashar [78] defined the polynomial $L_n^{(a,h)}(x)$ which is exactly the same as given by Tascano [133]. For $H_{2pk}(x;k)$ see Thakare and Karande's paper [130]. The polynomials $l_n^{(a)}(x)$, $A_n^{(a)}(x)$, $P_n^{(a,b;h)}(x)$, $M_n^k(x;a,b)$, $H_n^{(a,b)}(p,q;x)$, $Q_n(x;a,b,N)$, $S_n^{(v)}(x)$ and $J_n^{(a,b)}(x,k)$ are introduced by Tricomi [137], Srivastava [125], Tascano [134], Chatterjee [33], Khandekar [60], Hahn [59], Shivley [93] and Mandhekar & Thakare [64] respectively.

2.2 <u>Explicit Form</u> : Starting from (2.1.2) and using the formula

$$(2.2.1) \quad \Delta_{u,h}^{n} f(u) = \sum_{r=0}^{n} (-)^{n-r} \binom{n}{r} f(u+rh),$$

we can easily show that

$$(2.2.2)\quad G_n^{(u;h)}(x : (a); (b); (c) : (d); (e); (f))$$

$$= \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))} \sum_{r=0}^{n} (-)^r \binom{n}{r} \frac{\Gamma((a)u+(b)n+(c)+(a)hr)}{\Gamma((d)u+(e)+(d)hr)} x^{hr}.$$

In case $a_1h, \ldots, a_Ah, d_1h, \ldots, d_Dh$ are all positive integers, (2.2.2) can be written into the following hypergeometric form also

$$(2.2.3)\quad G_n^{(u;h)}(x: (a); (b); (c) : (d); (e); (f))$$

$$= \frac{\Gamma((d)u+(f)n+(e)}{\Gamma((d)u+(e))} \sum_{r=0}^{n} \frac{(-n)_r \ ((a)u+(b)n+(c))_{(a)hr}}{r! \ ((d)u+(e))_{(d)hr}} x^{hr}$$

$$= \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((d)u+(e))}$$

$$\times \ {}_{1+(a_1+\cdots+a_A)h}F_{(d_1+\cdots+d_D)h}\left[\begin{matrix} -n, \Delta((a)h; (a)u+(b)n+(c)); \\ \Delta((d)h; (d)u+(e) \qquad ; \end{matrix} \ Hx^h\right],$$

where for convenience

$$(2.2.4)\quad H = \frac{a_1^{a_1h} \cdots a_A^{a_Ah} \cdot h^{(a_1+\cdots+a_A)h-(d_1+\cdots+d_D)h}}{d_1^{d_1h} \cdots d_D^{d_Dh}},$$

also $\triangle((a)h;(a)u+(b)n+(c))$ stands for $\triangle(a_1h;a_1u+b_1n+c_1)$, ..., $\triangle(a_Ah;a_Au+b_An+c_A)$ and $\triangle((d)h;(d)u+(e))$ for $\triangle(d_1h;d_1u+e_1),\ldots,\triangle(d_Dh;d_Du+e_D)$. Here $\triangle(m;v)$ is taken to abbreviate the sequence of m factors v/m, $(v+1)/m,\ldots,(v+m-1)/m; m \geqslant 1$.

2.3 Linear Generating Functions : In this section for the positive integers $a_1h,\ldots,a_1h$ and $d_1h,\ldots,d_Dh$, we derive certain linear generating relations involving the polynomials $G_n^{(u;h)}(x)$ defined by (2.1.2).

(i) By use of $\underset{u,h}{\triangle} = \underset{u,h}{E} - 1$ and $\underset{u,h}{E} f(u) = f(u+h)$, we have

$$\sum_{n=0}^{\infty} \frac{(-)^n t^n}{n!} \underset{u,h}{\triangle^n} \left[\frac{\Gamma((a)u+(c))}{\Gamma((d)u+(e))} x^u\right] = \exp[-t(\underset{u,h}{E}-1)] \left[\frac{\Gamma((a)u+(c))}{\Gamma((d)u+(e))} x^u\right]$$

$$= e^t \sum_{r=0}^{\infty} (-)^r \frac{\Gamma((a)u+(a)hr+(c))}{\Gamma((d)u+(d)hr+(e))} t^r x^{u+hr}$$

$$= e^t \frac{\Gamma((a)u+(c))}{\Gamma((d)u+(e))} \sum_{r=0}^{\infty} \frac{((a)u+(c))_{(a)hr}}{((d)u+(e))_{(d)hr}} (-tx^h)^r ,$$

which with the help of (2.1.2), gives the generating relation

$$(2.3.1)\quad \sum_{n=0}^{\infty} \frac{t^n\,\Gamma((d)u+(e))}{n!\,\Gamma((d)u+(f)n+(e))}\, G_n^{(u;h)}(x:(a);(b);(c)-(b)n:(d);(e);(f))$$

$$= {}_{(a_1+\ldots+a_A)h}F_{(d_1+\ldots+d_D)h}\left[\begin{matrix}\triangle((a)h;(a)u+(c)) \;; \\ \triangle((d)h;(d)u+(c)) \;;\end{matrix}\; -Htx^h\right],$$

where H is given by (2.2.4).

(ii) We have

$$\sum_{n=0}^{\infty} \frac{(w)_n\, t^n}{n!\,\Gamma((d)u+(f)n+(e))}\, G_n^{(u;h)}(x:(a);(b);(c)-(b)n:(d);(e);(f))$$

$$= \frac{x^{-u}}{\Gamma((a)u+(c))} \sum_{n=0}^{\infty} \frac{(w)_n}{n!}(-t\,\triangle_{u,h})^n \left[\frac{\Gamma((a)u+(c))}{\Gamma((d)u+(e))}\, x^u\right].$$

In the above expression using the formula (2.2.1), we get another generating relation.

$$(2.3.2)\quad \sum_{n=0}^{\infty} \frac{(w)_n\, t^n\,\Gamma((d)u+(e))}{n!\,\Gamma((d)u+(f)n+(e))}\, G_n^{(u;h)}(x:(a);(b);(c)-(b)n:(d);(e);(f))$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n} (-)^r \binom{n}{r} \frac{(w)_n\,((a)u+(c))_{(a)hr}}{n!\,((d)u+(e))_{(d)hr}}\, x^{rh}\, t^n$$

$$= (1-t)^{-w}\, {}_{1+(a_1+\ldots+a_A)h}F_{(d_1+\ldots+d_D)h}\left[\begin{matrix} w, \triangle((a)h;(a)u+(c)) \;; \\ \triangle((d)h;(d)u+(e)) \;;\end{matrix}\; \frac{Htx^h}{t-1}\right].$$

(iii) In view of the definition (2.1.2), we can write

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} G_n^{(u;h)}(x:(a);(b);(c)-(b)n:(d);(e);(f))$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(c))n!} x^{-u}(-t)^n \underset{u,h}{\Delta} \left[\frac{\Gamma((a)u+(c))}{\Gamma((d)u+(e))} x^u \right]$$

$$= \sum_{n,r=0}^{\infty} \frac{\Gamma((d)u+(f)n+(f)r+(e))\ \Gamma((a)u+(c)+(a)hr)}{\Gamma((a)u+(c))\ \Gamma((d)u+(e)+(d)hr)\ n!\ r!} t^n(-tx^h)^r$$

which for the positive integers $f_1,\ldots,f_D$, gives us an interesting generating relation

$$(2.3.3)\quad \sum_{n=0}^{\infty} \frac{t^n}{n!} G_n^{(u;h)}(x:(a);(b);(c)-(b)n:(d);(e);(f))$$

$$= \sum_{n,r=0}^{\infty} \frac{((d)u+(e))_{(f)n+(f)r}\ ((a)u+(c))_{(a)hr}}{n!\ r!\ ((d)u+(e))_{(d)hr}} t^n(-x^h t)^r$$

$$= F\left[\begin{matrix} \Delta((f);(d)u+(e)):-:\Delta((a)h;(a)u+(c)); \\ -:-:\Delta((d)h;(d)u+(e)); \end{matrix}\ f_1^{f_1}\cdots f_D^{f_D}t, -f_1^{f_1}\cdots f_D^{f_D}Htx^h\right].$$

(iv) In a similar manner, by the use of (2.2.1), we see that

$$\sum_{n=0}^{\infty} \frac{(p)_n(v)_n t^n}{(w)_n\, n!\, \Gamma((d)u+(f)n+(e))} G_n^{(u;h)}(x:(a);(b);(c)-(b)n:(d);(e);(f))$$

$$= \sum_{n,r=0}^{\infty} \frac{(-)^r \, (p)_{n+r} \, (v)_{n+r} \, ((a)u+(c))_{(a)hr}}{n! \; r! \; \Gamma((d)u+(e)) \; (w)_{n+r} \; ((d)u+(e))_{(d)hr}} \, x^{hr} \, t^{n+r}$$

$$= \sum_{r=0}^{\infty} \frac{(p)_r \, (v)_r \, ((a)u+(c))_{(a)hr} \, (-tx^h)^r}{r! \; \Gamma((d)u+(e)) \; (w)_r \; ((d)u+(e))_{(d)hr}} \; {}_2F_1\left[\begin{matrix} p+r, \; v+r \; ; \\ w+r \; ; \end{matrix} \; t \right].$$

By the Eulers' transformation [85, (4), p.60], we have

$${}_2F_1\left[\begin{matrix} p+r, \; v+r \; ; \\ w+r \; ; \end{matrix} \; t \right] = (1-t)^{-p-r} \left[\begin{matrix} p+r, \; w-v \; ; \\ w+r \; ; \end{matrix} \; \frac{t}{t-1} \right].$$

Hence, we get the following generating function

$$(2.3.4) \quad \sum_{n=0}^{\infty} \frac{(p)_n \, (v)_n \, t^n}{(w)_n \, n! \, \Gamma((d)u+(f)n+(e))} \, G_n^{(u;h)}(x:(a);(b);(c)-(b)n:(d);(e);(f))$$

$$= \frac{(1-t)^{-p}}{\Gamma((d)u+(e))} \sum_{r,s=0}^{\infty} \frac{(p)_{r+s} \, (w-v)_s \, (v)_s \, ((a)u+(c))_{(a)hr}}{r! \; s! \; (w)_{r+s} \; ((d)u+(e))_{(d)hr}} \left(\frac{t}{t-1}\right)^s \left(\frac{x^h t}{t-1}\right)^r$$

$$= \frac{(1-t)^{-p}}{\Gamma((d)u+(e))} \; F\left[\begin{matrix} p: w-v; \; v, \Delta((a)h; \; (a)u+(c)) \; ; \\ w: \; - \; ; \quad \Delta((d)h; \; (d)u+(e)) \; ; \end{matrix} \; \frac{t}{t-1}, \; \frac{Htx^h}{t-1} \right].$$

For, w=v (2.3.4) reduces into (2.3.2).

(v) By using (2.2.1), we can easily show that

$$(2.3.5)\quad \sum_{n=0}^{\infty} \frac{(w)_n\, t^n}{n!\,\Gamma((d)u+(f)n+(f)m+(e))}\, G_{m+n}^{(u;h)}(x:(a);(b);(c)-(b)n:(d);(e);(f))$$

$$= \sum_{n,r=0}^{\infty} \sum_{s=0}^{m} \frac{(w+r)_n\, (w)_r\, (-m)_s\, ((a)u+(b)m+(c))_{(a)hr+(a)hs}\, x^{hs}(-tx^h)^r}{n!\; r!\; s!\; \Gamma((d)u+(e))\; ((d)u+(e))_{(a)hr+(a)hs}}$$

$$= \frac{(1-t)^{-w}}{\Gamma((d)u+(e)}\, F\left[\begin{matrix}\triangle((a)h; & (a)u+(b)m+(c)):-m; & w\ ; \\ \triangle((d)h; & (d)u+(e)) \quad :-\ ; & -\ ;\end{matrix}\ Hx^h, \frac{Htx^h}{t-1}\right].$$

Particular Cases : By giving different values to parameters in the above results, a number of known and unknown generating relations can be obtained. Some of them are given below :

(i) The substitution $A=D=1$, $a_1=0$, $b_1=0$, $c_1=1$, $d_1=1$, $e_1=1$, $f_1=1$, $h_1=1$, $u=a$ and $v=a+1$, in the above equations (2.3.1), (2.3.2), (2.3.4), (2.3.5) and use of the result (2.1.3) for $h=1$, we get the following generating relation for Laguerre polynomials :

$$(2.3.6)\quad \sum_{n=0}^{\infty} \frac{t^n}{(1+a)_n}\, L_n^{(a)}(x) = e^t\ {}_0F_1(-;\ 1+a;-xt);\ [85,\ p.201]\ ,$$

$$(2.3.7)\quad \sum_{n=0}^{\infty} \frac{(w)_n\, t^n}{(1-a)_n}\, L_n^{(a)}(x) = (1-t)^{-w}\ {}_1F_1\left[\begin{matrix} w\ ; \\ 1+a\ ;\end{matrix}\ \frac{-xt}{1-t}\right];\ [85,p.202]\ ,$$

$$(2.3.8)\quad \sum_{n=0}^{\infty} \frac{(p)_n}{(w)_n}\, L_n^{(a)}(x)\, t^n = (1-t)^{-p}\ \phi_1[p:w-a-1;w;\frac{t}{t-1}, \frac{xt}{t-1}]\ [119,p.13]$$

and

$$(2.3.9) \quad \sum_{n=0}^{\infty} \binom{m+n}{n} \frac{(w)_n}{(a+m+1)_n} L_{m+n}^{(a)}(x)\ t^n$$

$$= \binom{a+m}{m} (1-t)^{-w} \ \phi_2 \left[-m, w; a+1; x, \frac{xt}{t-1}\right] ; \qquad [119, \text{ p.}132].$$

(ii) Taking $A=D=2$, $a_1=1$, $a_2=1$, $b_1=1$, $b_2=0$, $c_1=1+a+b$, $c_2=p$, $d_1=d_2=1$, $e_1=1+a$, $e_2=q$, $h=1$ and $p=s$ in (2.3.1), (2.3.2), (2.3.4), (2.3.5), and using the result (2.1.9) after taking limit u tending to 0, we get the following generating relations involving generalized Rice's polynomials:

$$(2.3.10) \quad \sum_{n=0}^{\infty} \frac{t^n}{(a+1)_n} H_n^{(a,b-n)}(p,q;x) = e^t \ {}_2F_2\left[\begin{matrix} p, & a+b+1 \ ; \\ q, & a+1 \ ; \end{matrix} \ -xt\right],$$

$$(2.3.11) \quad \sum_{n=0}^{\infty} \frac{(w)_n \ t^n}{(1+a)_n} H_n^{(a,b-n)}(p,q;x)$$

$$= (1-t)^{-w} \ {}_3F_2 \left(w, p, a+p+1; q, a+1 \ ; \frac{xt}{t-1}\right),$$

$$(2.3.12) \quad \sum_{n=0}^{\infty} \frac{(s)_n \ (v)_n \ t^n}{(w)_n \ (1+a)_n} H_n^{(a,b-n)}(p,q;x)$$

$$= (1-t)^{-s} \ F\left[\begin{matrix} s : & w-v; & v, & p, & a+b+1 \ ; \\ w : & - \ ; & & q, & a+1 \ ; \end{matrix} \ \frac{t}{t-1}, \frac{xt}{t-1}\right]$$

and

$$= e^{t} \sum_{n=0}^{\infty} \frac{((a)u+(c))_{(a)hn} \; ((a')\, v+(c'))_{(a')\, kn}}{((d)u+(e))_{(d)hn} \; ((d')\, v+(e'))_{(d')\, kn}} \cdot \frac{(x^{h}\, y^{k}\, t)^{n}}{n!}$$

$$\times \sum_{r=0}^{\infty} \frac{((a)u+(a)hn+(c))_{(a)hr}}{((d)u+(d)hn+(e))_{(d)hr}} \cdot \frac{(-tx^{h})^{r}}{r!}$$

$$\times \sum_{s=0}^{\infty} \frac{((a')\, v+(a')\, kn+(c'))_{(a')\, ks}}{((d')\, v+(d')\, kn+(e'))_{(d')\, ks}} \cdot \frac{(-ty^{k})^{s}}{s!}$$

$$= e^{t} \sum_{n=0}^{\infty} \frac{((a)u+(c))_{(a)hn} \; ((a')\, v+(c'))_{(a')\, kn}}{((d)u+(e))_{(d)hn} \; ((d')\, v+(e'))_{(d')\, kn}} \cdot \frac{(x^{h}\, y^{k}\, t)^{n}}{n!}$$

$$\times \; {}_{(a_1+\dots+a_A)h}F_{(d_1+\dots+d_D)h}\left[\begin{matrix}\triangle((a)h;\,(a)u+(a)hn+(c)) \;; \\ \triangle((d)h;\,(d)u+(d)hn+(e)) \;;\end{matrix}\; -Htx^{h}\right]$$

$$\times \; {}_{(a'_1+\dots+a'_{A'})k}F_{(d'_1+\dots+d'_{D'})k}\left[\begin{matrix}\triangle((a')k;\,(a')v+(a')kn+(c')) \;; \\ \triangle((d')k;\,(d')v+(d')kn+(e')) \;;\end{matrix}\; -kty^{k}\right],$$

where H is given by (2.2.4) and

$$(2.4.2) \quad K = \frac{a_1'^{\,a'_1 k} \cdots a_{A'}'^{\,a'_{A'} k} \cdot k^{(a'_1+\dots+a'_{A'})k-(d'_1+\dots+d'_{D'})k}}{d_1'^{\,d'_1 k} \cdots d_{D'}'^{\,d'_{D'} k}} .$$

(ii) Starting from (2.1.2) and using (2.2.1), we consider

$$\sum_{n=0}^{\infty} \frac{(v)_n \; t^{n} \; \Gamma((a)u+(c)) \; \Gamma((a')\, v+(c'))}{n! \; \Gamma((d)u+(f)n+(e)) \; \Gamma((d')\, v+(f')\, n+(e'))}$$

$$\times\ G_n^{(u;h)}(x:(a);(b);(c)-(b)n:(d);(e);(f))$$

$$\times\ G_n^{(v;k)}(y:(a');(b');(c')-(b')n:(d');(e');(f'))$$

$$=\sum_{n,p=0}^{\infty}\frac{(-)^n\,(w)_{n+p}\,\Gamma((a)u+(a)hp+(c))}{n!\;p!\;\Gamma((d)u+(d)hp+(e))}\,t^{n+p}\,y^{-v}\,x^{ph}$$

$$\times\ \Delta_{v,k}^{n+p}\left[\frac{\Gamma((a')v+(c'))}{\Gamma((d')v+(e'))}\,y^{v}\right].$$

Again using (2.2.1) and after some simplification, we get the following bilateral generating function

$$(2.4.3)\quad \sum_{n=0}^{\infty}\frac{(w)_n\,G_n^{(u;h)}(x:(a);(b);(c)-(b)n:(d);(e);(f))}{n!\;\Gamma((d)u+(f)n+(e))\;\Gamma((d')v+(f')n+(e'))}$$

$$\times\ G_n^{(v,k)}(y:(a');(b');(c')-(b')n:(d');(e');(f'))\,t^n$$

$$=\frac{(1-t)^{-w}}{\Gamma((d)u+(e))\,\Gamma((d')v+(e'))}\,F^{(3)}\left[\begin{matrix}w::\Delta((a)h;(a)u+(c))\;; \\ -::\Delta((d)h;(d)u+(e))\;;\end{matrix}\right.$$

$$\left.\begin{matrix}-;\Delta((a')k;(a')v+(c'));-;-;-; \\ -;\Delta((d')k;(d')v+(e')):-;-;-;\end{matrix}\quad \frac{HKtx^h y^k}{1-t},\ \frac{Htx^h}{t-1},\ \frac{Kty^k}{t-1}\right],$$

where

$$F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'')\;; \\ (e)::(g);(g');(g''):(h);(h');(h'')\;;\end{matrix}\quad x,y,z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \cdot \frac{((a))_{m+n+p} \; ((b))_{m+n} \; ((b'))_{n+p} \; ((b''))_{p+m}}{((e))_{m+n+p} \; ((g))_{m+n} \; ((g'))_{n+p} \; ((g''))_{p+m}}$$

$$X \quad \frac{((c))_m \; ((c'))_n \; ((c''))_p \; x^m \; y^n \; z^p}{((h))_m \; ((h'))_n \; ((h''))_p \; m! \; n! \; p!}$$

(iii) In generating relation (2.3.2) after replacing w by $w+v$ and multiplying both sides by $\Gamma((p)+v)\,y^v / \Gamma((q)+v)$, operating by the operator $\triangle_v^m$, and using (2.2.1) for h=1, we get after simplification and taking v=0 finally

$$(2.4.4) \quad \sum_{n=0}^{\infty} \frac{(w)_n \; t^n}{n! \; \Gamma((d)u+(f)n+(e))} \; {}_{P+2}F_Q \left[\begin{matrix} -m, \; w+n, \; (p) \; ; \\ (q) \; ; \end{matrix} \; y \right]$$

$$= \frac{(1-t)^{-w}}{\Gamma((d)u+(e))} \; F \left[\begin{matrix} w \; : -m, (p) ; \triangle((a)h; (a)u+(c)) ; \\ - \; : \quad (q) ; \triangle((d)h; (d)u+(e)) ; \end{matrix} \; \frac{y}{1-t} \; , \; \frac{tHx^h}{t-1} \right].$$

(iv) Multiplying both sides of equation (2.3.1) by $\Gamma((p)+v) / \Gamma((q)+v)$ and replacing t by tE_v, we have

$$\sum_{r=0}^{\infty} \frac{((a)u+(c))_{(a)hr} \; x^{hr} (-t)^r}{\Gamma((d)u+(e)) \; ((d)u+(e))_{(d)hr} \; r!} \; E_v^r \left[\frac{\Gamma((p)+v)}{\Gamma((q)+v)} \right]$$

$$= \sum_{n,r=0}^{\infty} \frac{(-)^r \; t^{n+r}}{\Gamma((d)u+(f)n+(e)) \; r!} \; E_v^{n+r} \left[\frac{\Gamma((p)+v)}{\Gamma((q)+v)} \right]$$

$$\times\; G_n^{(u;h)}(x:(a);(b);(c)-(b)n:(d);(e);(f)),$$

which after some simplification and putting $v=0$, gives

$$(2.4.5)\quad {}_{P+(a_1+\ldots+a_A)h}F_{Q+(d_1+\ldots+d_D)h}\left[\begin{matrix}(p),\Delta((a)h;(a)u+(c));\\(q),\Delta((d)h;(d)u+(e));\end{matrix}\; -Htx^h\right]$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma((d)u+(e))\;((p))_n\; t^n}{\Gamma((d)u+(f)n+(e))((q))_n n!}\;{}_PF_Q\left[\begin{matrix}(p)+n\;;\\(q)+n\;;\end{matrix}\; -t\right]$$

$$\times\; G_n^{(u;h)}(x:(a);(b);(c)-(b)n:(d);(e);(f)).$$

In particular, if in (2.4.5) we replace (p) by $p,\Delta((d)h;(d)u+(e))$; (q) by $\Delta((a)h;(a)u+(c))$ and Ht by t, we obtain an interesting relation

$$(2.4.6)\quad (1+x^h t)^{-p}=\sum_{n=0}^{\infty}\frac{\Gamma((d)u+(d)hn+(e))\;(p)_n\; t^n}{n!\,\Gamma((d)u+(f)n+(e))\;((a)u+(c))_{(a)hn}}$$

$$\times\; G_n^{(u;h)}(x:(a);(b);(c)-(b)n:(d);(e);(f))$$

$$\times\; {}_{1+(d_1+\ldots+d_D)h}F_{(a_1+\ldots+a_A)h}\left[\begin{matrix}p+n,\Delta((d)h;(d)u+(d)hn+(e));\\\Delta((a)h;(a)u+(a)hn+(c));\end{matrix}\;\frac{-t}{H}\right].$$

<u>Particular Cases:</u> By giving different values to parameters in the above result of section 2.4, a number of known and unknown bilinear and bilateral generating functions can be obtained. Here we shall quote only a few of them :

(i) The substituting in (2.4.1), (2.4.3), (2.4.4), (2.4.5) and (2.4.6) $A=D=1$, $a_1=0$, $b_1=0$, $c_1=d_1=e_1=f_1=1$, $u=a$, $h=1$ and $A'=D'=1$, $a'_1=b'_1=0$, $c'_1=d'_1=e'_1=f'_1=1$, $v=b$, $k=1$, we get

$$(2.4.7)\quad \sum_{n=0}^{\infty} \frac{n!}{(a+1)_n\ (b+1)_n}\, L_n^{(a)}(x)\, L_n^{(b)}(y)\, t^n$$

$$= e^t \sum_{n=0}^{\infty} \frac{(xyt)^n}{n!\,(a+1)_n\ (b+1)_n}\ {}_0F_1\left[\begin{matrix} - \ ; \\ a+n+1\ ; \end{matrix}\ -xt\right] {}_0F_1\left[\begin{matrix} - \ ; \\ b+n+1\ ; \end{matrix}\ -yt\right];$$

[119, (15), p.134] ,

$$(2.4.8)\quad \sum_{n=0}^{\infty} \frac{n!\ (w)_n}{(a+1)_n\ (b+1)_n}\ L_n^{(a)}(x)\, L_n^{(b)}(y)\, t^n$$

$$=(1-t)^{-w}\, F^{(3)}\left[\begin{matrix} w\ ::\ -;\ -;\ -\ :\ -;\ -;\ -; \\ -\ ::\ a+1;\ -;\ b+1\ :\ -;\ -;\ -; \end{matrix}\ \frac{xyt}{1-t},\ \frac{xt}{t-1},\ \frac{yt}{t-1}\right];$$

[119, (9), p.132],

$$(2.4.9)\quad \sum_{n=0}^{\infty} \frac{(w)_n}{(a+1)_n}\, L_n^{(a)}(x)\ {}_{P+2}F_Q\left[\begin{matrix} -m,\ w+n,\ (p)\ ; \\ (q)\ ; \end{matrix}\ y\right] t^n$$

$$=(1-t)^{-w}\, F\left[\begin{matrix} w\ :\ -m,\ (p)\ ;\ -\ ; \\ -\ :\ (q)\ ;\ a+1\ ; \end{matrix}\ \frac{y}{(1-t)},\ \frac{-xt}{1-t}\right],$$

$$(2.4.10)\quad \sum_{n=0}^{\infty} \frac{((p))_n}{((q))_n\ (a+1)_n}\, L_n^{(a)}(x)\ {}_{\cancel{A}}F_{\cancel{B}}\left[\begin{matrix} (p)+n\ ; \\ (q)+n\ ; \end{matrix}\ -t\right] t^n$$

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$$= {}_pF_{Q+1}\left[\begin{matrix} & (p) \; ; & \\ a+1, & (q) \; ; & \end{matrix} -xt\right]$$

and

$$(2.4.11) \quad (1+xt)^{-p} = \sum_{n=0}^{\infty} (p)_n \, L_n^{(a)}(x) \; t^n \; {}_2F_1(a+n+1,\, p+n; -; -t) .$$

0/

(ii) Now by taking $A=D=A'=D'=1$, $a_1=b_1=d_1=e_1=f_1$ $=a'_1=b'_1=d'_1=f'_1=h=k=1$, $c_1=b+1$, $c'_1=d+1$, $u=a$, $v=c$, $x=(1-x)/2$, $y=(1-y)/2$ in above equations of section 2.4, we get following bilinear and bilateral generating relation involving Jacobi polynomials :

$$(2.4.12) \quad \sum_{n=0}^{\infty} \frac{n! \; t^n}{(a+1)_n \; (b+1)_n} P_n^{(a;b-n)}(x) \; P_n^{(c;d-n)}(x)$$

$$= e^t \sum_{n=0}^{\infty} \frac{(a+b+1)_n \; (c+d+1)_n}{(a+1)_n \; (c+1)_n} \left[(1-x)(1-y)t/4\right]^n$$

$$\times \; {}_1F_1\left[\begin{matrix} a+b+n+1 \; ; \\ a+n+1 \; ; \end{matrix} \; \frac{t(x-1)}{2}\right] {}_1F_1\left[\begin{matrix} c+d+n+1 \; ; \\ c+n+1 \; ; \end{matrix} \; \frac{t(y-1)}{2}\right],$$

$$(2.4.13) \quad \sum_{n=0}^{\infty} \frac{n! \; (w)_n \; t^n}{(a+1)_n \; (b+1)_n} P_n^{(a;b-n)}(x) \; P_n^{(c;d-n)}(y) = (1-t)^{-w}$$

$$\times \; F^{(3)}\left[\begin{matrix} w \; :: \; a+b+1; -; & c+d+1 \; : -; -; -; \\ - \; :: \; a+1; -; & c+1 \; : -; -; -; \end{matrix} \; \frac{t(1-x)(1-y)}{4(1-t)}, \frac{t(1-x)}{2(t-1)}, \frac{t(1-y)}{2(t-1)}\right],$$

$$(2.4.14)\quad \sum_{n=0}^{\infty} \frac{(w)_n\, t^n}{(a+1)_n} P_n^{(a,b-n)}(x)\ {}_{P+2}F_Q\left[\begin{matrix}-m,\ w+n,\ (p)\ ; \\ (q)\ ;\end{matrix}\ y\right]$$

$$= (1-t)^{-w}\ F\left[\begin{matrix} w : -m, (p)\ ;\ a+b+1\ ; \\ - :\quad (q)\ ;\quad a+1\ ;\end{matrix}\ \frac{y}{(1-t)},\ \frac{t(1-x)}{2(t-1)}\right],$$

$$(2.4.15)\quad \sum_{n=0}^{\infty} \frac{((p))_n\, t^n}{(a+1)_n\,((q))_n} P_n^{(a;b-n)}(x)\ {}_PF_Q\left[\begin{matrix}(p)+n\ ; \\ (q)+n\ ;\end{matrix}\ -t\right]$$

$$= {}_{P+1}F_{Q+1}\left[\begin{matrix} a+b+1,\ (p)\ ; \\ a+1,\ (q)\ ;\end{matrix}\ \frac{t(x-1)}{2}\right]$$

and

$$(2.4.16)\quad \left[1+\frac{(1-x)t}{2}\right]^{-p} = \sum_{n=0}^{\infty} \frac{(p)_n\, t^n}{(a+b+1)_n} P_n^{(a;b-n)}(x)\ {}_2F_1\left[\begin{matrix}p+n, a+n+1; \\ a+b+n+1;\end{matrix}\ -t\right].$$

(iii) If in (2.4.1) and (2.4.3) $A=D=A'=D'=1$,

$a_1=0$, $b_1=0$, $c_1=d_1=e_1=f_1=h=a_1'=b_1'=e_1'=f_1'=k=1$, $c_1'=c+1$, $u=a$,

$v=b$ and $y=(1-y)/2$, then we can easily show that

$$(2.4.17)\quad \sum_{n=0}^{\infty} \frac{n!\ t^n}{(a+1)_n\ (b+1)_n} L_n^{(a)}(x)\ P_n^{(b;c-n)}(y)$$

$$= e^t \sum_{n=0}^{\infty} \frac{(b+c+1)_n}{(a+1)_n\ (b+1)_n\ n!} [x(1-y)t/2]^n$$

$$X\ {}_0F_1(-;\ a+n+1; -xt)\ {}_1F_1(b+c+n+1;\ b+n+1;\ t(y-1)/2)$$

and

$$(2.4.18)\quad \sum_{n=0}^{\infty} \frac{n!\,(w)_n}{(a+1)_n\,(b+1)_n}\, L_n^{(a)}(x)\, P_n^{(b,c-n)}(y)\, t^n$$

$$= (1-t)^{-w}\, F^{(3)}\left[\begin{matrix} w::-;-;\ b+c+1\ :-;-;-; \\ -::a+1;-;\ b+1\ :-;-;-; \end{matrix}\ \frac{tx(1-y)}{2(1-t)},\ \frac{tx}{t-1},\ \frac{t(1-y)}{2(t-1)}\right].$$

2.5 Operational Formulae : In this section we shall derive the following operational formulae :

$$(2.5.1)\quad \frac{\Gamma((d)u+(e))}{\Gamma((a)u+(b)n+(c))} \sum_{r=0}^{n} (-)^{n-r} \binom{n}{r} \frac{\Gamma((a)u+(b)n+(c)+(a)hr)\ x^{hr}}{\Gamma((d)u+(f)n-(f)r+(e)+(d)hr)}$$

$$\times\ G_{n-r}^{(u+hr;\,h)}(x:(a);(b);(c)+(b)r:(d);(e);(f))\ \Delta_{u,h}^{r}\,[f(u)]$$

$$= \frac{\Gamma((d)u+(e))}{\Gamma((a)u+(b)n+(c))}\,(x^h E_{u,h}-1)^n \left[\frac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))}\, f(u)\right]$$

$$= \prod_{j=1}^{n} \left[x^h\, \frac{(a_1u+b_1n+c_1-n-1+a_1hj+j)\ \Gamma(a_1u+b_1n+c_1+a_1h-n-1+j)}{\Gamma(a_1u+b_1n+c_1-n+j)}\right.$$

$$\left.\times\ \frac{\Gamma(a_2u+b_2n+c_2+a_2h)\ldots\Gamma(a_Au+b_An+c_A+a_Ah)\,\Gamma((d)u+(e))}{\Gamma(a_2u+b_2n+c_2)\ldots\Gamma(a_Au+b_An+c_A)\,\Gamma((d)u+(e)+(d)h)}\, E_{u,h} - 1\right] f(u).$$

To prove consider

$$(2.5.2)\quad R_n[f(u)] = \frac{x^{-u}\,\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))}\, \Delta_{u,h}^{n} \left[\frac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))}\, x^u f(u)\right].$$

By making use of the formula

(2.5.3) $$\underset{u,h}{\Delta^n}[f(u)\ g(u)] = \sum_{r=0}^{n} \binom{n}{r} \underset{u,h}{\Delta^{n-r}} [f(u+hr)] \underset{u,h}{\Delta^r}[g(u)] ,$$

on the R.H.S. of (2.5.2), we get

$$R_n\ [f(u)] = \frac{x^{-u}\ \Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))} \sum_{r=0}^{n} \binom{n}{r}$$

$$\times \underset{u,h}{\Delta^{n-r}} \left[\frac{\Gamma((a)u+(b)n+(c)+(a)hr)}{\Gamma((d)u+(e)+(d)hr)}\ x^{u+hr} \right] \underset{u,h}{\Delta^r}\ [f(u)] ,$$

which in view of the definition (2.1.2), reduces to

(2.5.4) $$R_n\ [f(u)] = \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))}$$

$$\times \sum_{r=0}^{n} (-)^{n-r} \binom{n}{r} \frac{\Gamma((a)u+(b)n+(c)+(a)hr)}{\Gamma((d)u+(f)n-(f)r+(e)+(d)hr)}\ x^{hr}$$

$$\times\ G_{n-r}^{(u+hr;h)}\ (x : (a); (b); (c)+(b)r : (d); (e); (f)) \underset{u,h}{\Delta}\ [f(u)] .$$

Further consider the n th difference

$$\underset{u,h}{\Delta^n}\ [x^u\ g(u)] = \sum_{r=0}^{n} (-)^{n-r} \binom{n}{r}\ x^{u+hr}\ g(u+hr)$$

$$= x^u \sum_{r=0}^{n} (-)^{n-r} \binom{n}{r}\ x^{hr} \underset{u,h}{E^r}\ [g(u)]$$

$$= x^u \; (x^h \underset{u,h}{E} - 1)^n \, [g(u)] \; .$$

Now substituting $g(u) = \dfrac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))} f(u)$, and using (2.5.2), we obtain

$$(2.5.5) \quad R_n[f(u)] = \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))}$$

$$\times \; (x^h \underset{u,h}{E} -1)^n \, \left[\frac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))} f(u) \right].$$

From (2.5.3), we have

$$\underset{u,h}{\Delta^n} [\, \Gamma(a_1u+b_1n+c_1) \; g(u) \; f(u) \,]$$

$$= \sum_{r=0}^{n} \binom{n}{r} \underset{u,h}{\Delta^{n-r}} [a_1u+b_1n+c_1-1+a_1hr] \underset{u,h}{\Delta^r} [\, \Gamma(a_1u+b_1n+c_1-1)\, g(u) f(u)]$$

$$= (a_1u+b_1n+c_1-1+a_1hn) \underset{u,h}{\Delta^n} [\, \Gamma(a_1u+b_1n+c_1-1)\, g(u) f(u)]$$

$$+ \; na_1h \underset{u,h}{\Delta^{n-1}} [\, \Gamma(a_1u+b_1n+c_1-1)\, g(u) f(u)],$$

as $\underset{u,h}{\Delta} [a_1u+b_1n+c_1-1+a_1h(n-1)] = a_1h.$

Thus we have,

$$\underset{u,h}{\Delta^n} [\, \Gamma(a_1u+b_1n+c_1) \; g(u) \; f(u) \,]$$

$$= [(a_1u+b_1n+c_1-1+a_1hn) \underset{u,h}{\Delta} + na_1h] \underset{u,h}{\Delta^{n-1}} [\, \Gamma(a_1u+b_1n+c_1-1)\, g(u) f(u) \,]$$

$$- [(a_1u+b_1n+c_1-1+a_1hn)\,E_{u,h} - (a_1u+b_1n+c_1-1)\,]$$

$$\times \underset{u,h}{\Delta}^{n-1}\,[\,\Gamma(a_1u+b_1n+c_1-1)\;g(u)\;f(u)\,]\,,$$

which on iteration yields

$$(2.5.6)\quad \underset{u,h}{\Delta}^{n}[\,\Gamma(a_1u+b_1n+c_1)\,g(u)\,f(u)\,] = \prod_{j=0}^{n-1}[(a_1u+b_1n+c_1+a_1hn$$

$$-a_1hj-j-1)\,E_{u,h} -(a_1u+b_1n+c_1-j-1)]\;[\,\Gamma(a_1u+b_1n+c_1-n)\,g(u)\,f(u)\,].$$

On putting $g(u) = \dfrac{\Gamma(a_2u+b_2n+c_2)\ldots\Gamma(a_Au+b_An+c_A)}{\Gamma((d)u+(e))}\,x^u$

in (2.5.6), we have

$$\underset{u,h}{\Delta}^{n}\,[\,\frac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))}\,x^u\;f(u)\,]$$

$$= \prod_{j=0}^{n-1}\,[(a_1u+b_1n+c_1+a_1hn-a_1hj-j-1)\,E_{u,h} -(a_1u+b_1n+c_1-j-1)\,]$$

$$\times\,[\,\frac{\Gamma(a_1u+b_1n+c_1-n)\;\Gamma(a_2u+b_2n+c_2)\ldots\Gamma(a_Au+b_An+c_A)}{\Gamma((d)u+(e))}\,x^u\;f(u)\,]$$

$$= \prod_{j=0}^{n-2}\,[(a_1u+b_1n+c_1+a_1hn-a_1hj-j-1)\,E_{u,h} -(a_1u+b_1n+c_1-j-1)\,]$$

$$\times\,\frac{\Gamma(a_1u+b_1n+c_1+a_1h-n+1)\;\Gamma(a_2u+b_2n+c_2)\ldots\Gamma(a_Au+b_An+c_A)}{\Gamma((d)u+(e))}\,x^u$$

$$\times \left[x^{h} \frac{\Gamma(a_1u+b_1n+c_1+a_1h-n)\ \Gamma(a_1u+b_1n+c_1+a_1h-n)\ \Gamma((d)u+(e))}{\Gamma(a_1u+b_1n+c_1-n+1)\ \Gamma((d)u+(e)+(d)h)}\right.$$

$$\left.\times \frac{\Gamma(a_2u+b_2n+c_2+a_2h)\ldots\Gamma(a_Au+b_An+c_A+a_Ah)}{\Gamma(a_2u+b_2n+c_2)\ldots\Gamma(a_Au+b_An+c_A)} E_{u,h} - 1\right] f(u) .$$

Repeating the above process $n-1$ times, we get

$$\mathop{\Delta}_{u,h}^{n} \left[\frac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))} x^{u} f(u)\right] = \frac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))} x^{u}$$

$$\times \prod_{j=1}^{n} \left[x^{h} \frac{(a_1u+b_1n+c_1-n-1+a_1hj+j)\ \Gamma(a_1u+b_1n+c_1+a_1h-n-1+j)}{\Gamma(a_1u+b_1n+c_1-n+j)\ \Gamma((d)u+(e)+(d)h)}\right.$$

$$\left.\times \frac{\Gamma((d)u+(e))\ \Gamma(a_2u+b_2n+c_2+a_2h)\ldots\Gamma(a_Au+b_An+c_A+a_Ah)}{\Gamma(a_2u+b_2n+c_2)\ldots\Gamma(a_Au+b_An+c_A)} E_{u,h} -1\right] f(u).$$

Therefore

$$(2.5.7)\quad R_n\left[f(u)\right] = \frac{\Gamma((d)u+(f)n+(e)}{\Gamma((d)u+(e))} \times$$

$$\times \prod_{j=1}^{n} \left[x^{h} \frac{(a_1u+b_1n+c_1-n-1+a_1hj+j)\ \Gamma(a_1u+b_1n+c_1+a_1h-n-1+j)}{\Gamma(a_1u+b_1n+c_1-n+j)}\right.$$

$$\left.\times \frac{\Gamma(a_2u+b_2n+c_2+a_2h)\ldots\Gamma(a_Au+b_An+c_A+a_Ah)\ \Gamma((d)u+(e))}{\Gamma(a_2u+b_2n+c_2)\ldots\Gamma(a_Au+b_An+c_A)\ \Gamma((d)u+(e)+(d)h)} E_{u,h} -1\right] f(u).$$

Combining (2.5.4), (2.5.5) and (2.5.7), we get the required result (2.5.1).

For $f(u)=1$, (2.5.7) gives us

$$(2.5.8)\quad G_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f)) = \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((d)u+(e))}$$

$$\times\left[1-\frac{(a_1u+b_1n+c_1-n-1+a_1hj+j)\ \Gamma(a_1u+b_1n+c_1+a_1h-n-1+j)}{\Gamma(a_1u+b_1n+c_1-n+j)}\right.$$

$$\left.\times\frac{\Gamma(a_2u+b_2n+c_2+a_2h)\ldots\Gamma(a_Au+b_An+c_A+a_Ah)\ \Gamma((d)u+(e))}{\Gamma(a_2u+b_2n+c_2)\ldots\Gamma(a_Au+b_An+c_A)\ \Gamma((d)u+(e)+(d)h)}\,E_{u,h}\right].1\,.$$

2.6 <u>Recurrence and Other Relations</u> : Substituting $f(u) = a_1u+b_1n+c_1$ in the operational formula (2.5.4), we have

$$x^{-u}\underset{u,h}{\overset{n}{\Delta}}\left[\frac{\Gamma(a_1u+b_1n+c_1)\ \Gamma(a_2u+b_2n+c_2)\ldots\Gamma(a_Au+b_An+c_A)}{\Gamma((d)u+(e))}\,x^u\right]$$

$$=\sum_{r=0}^{n}(-)^{n-r}\binom{n}{r}\frac{\Gamma((a)u+(b)n+(c)+(a)hr)}{\Gamma((d)u+(f)n-(f)r+(e)+(d)hr)}\,x^{hr}$$

$$\times\, G_{n-r}^{(u+hr;h)}(x:(a);(b);(c)+(b)r:(d);(e);(f))\underset{u,h}{\overset{r}{\Delta}}[a_1u+b_1n+c_1]\,,$$

which on using (2.1.2) gives the recurrence relation

$$(2.6.1)\quad G_n^{(u;h)}(x:(a);(b);c_1+1,c_2,\ldots,c_A:(d);(e);(f))$$

$$= G_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f))$$

$$- a_1 hnx^h \frac{\Gamma((a)u+(a)h+(b)n+(c)) \ \Gamma((d)u+(f)n+(e))}{(a_1u+b_1n+c_1)\Gamma((a)u+(b)n+(c))\Gamma((d)u+(d)h+(f)n-(f)+(e))}$$

$$\times \ G_{n-1}^{(u+h;h)} (x : (a); (b); (b)+(c) \ : (d); (e); (f)).$$

Again if we set $f(u)=d_1u+e_1-1$ in (2.5.4), we observe that

$$x^{-u} \Delta_{u,h}^{n} \left[\frac{\Gamma((a)\{u-1\}+(b)n+(a)+(c)) \ x^u}{\Gamma(d_1\{u-1\}+d_1+e_1-1)\Gamma(d_2\{u-1\}+d_2+e_2)\ldots\Gamma(d_D\{u-1\}+d_D+e_D)} \right]$$

$$= \frac{(-)^n \ \Gamma((a)u+(b)n+(c)) \ (d_1u+e_1-1)}{\Gamma((d)u+(f)n+(e))} G_n^{(u;h)} (x: (a); (b); (c) : (d); (e); (f))$$

$$- (-)^n d_1 hn \frac{\Gamma((a)u+(b)n+(c)+(a)h) \ x^h}{\Gamma((d)u+(f)n-(f)+(e)+(d)h)}$$

$$\times \ G_{n-1}^{(u+h;h)} (x : (a); (b); (b)+(c) \ : (d); (e); (f)) .$$

Using (2.1.2) on the R.H.S., we get another recurrence relation

$$(2.6.2) \ G_n^{(u-1;h)} (x: (a); (b); (a)+(c) : (d); d_1+e_1-1, d_2+e_2, \ldots, d_D+e_D; (f))$$

$$= \frac{(d_1u+e_1-1)}{(d_1u+f_1n+e_1-1)} G_n^{(u;h)} (x : (a); (b); (c) \ : (d); (e); (f))$$

$$- \frac{d_1 hn \Gamma((a)u+(b)n+(c)+(a)h) \ \Gamma((d)u+(f)n+(e)) \ x^h}{(d_1u+f_1n+e_1-1)\Gamma((a)u+(b)n+(c))\Gamma((d)u+(f)n-(f)+(e)+(d)h)}$$

$$\times \; G_{n-1}^{(u+h;h)}(x:(a);(b);(b)+(c):(d);(e);(f)) .$$

From (2.1.2) it follows that

$$G_n^{(u;h)}(x:(a);(b);(c);(d);(e);(f)) = \frac{(-)^n \Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))}$$

$$\times \; x^{-u-b_1n/a_1-c_1/a_1} \underset{u,h}{\Delta^n} \left[\frac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))} x^{u+b_1n/a_1+c_1/a_1} \right] ,$$

On differentiating it with respect to 'x', we get

$$DG_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f))$$

$$= \frac{(-)^{n+1}(a_1u+b_1n+c_1)\;\Gamma((d)u+(f)n+(e))\;x^{-u-1}}{a_1\;\Gamma((a)u+(b)n+(c))} \underset{u,h}{\Delta^n}\left[\frac{\Gamma((a)u+(b)n+(c)}{\Gamma((d)u+(e))} x^u\right]$$

$$+ \frac{(-)^{n+1}\Gamma((d)u+(f)n+(e))\;x^{-u}}{\Gamma((a)u+(b)n+(c))} \underset{u,h}{\Delta^n}\left[\frac{(a_1u+b_1n+c_1)\Gamma((a)u+(b)n+(c))}{a_1\;\Gamma((d)u+(e))} x^{u-1}\right].$$

After simplification, we get the differential recurrence relation

$$(2.6.3) \quad a_1xDG_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f))$$

$$=(a_1u+b_1n+c_1)\left[G_n^{(u;h)}(x:(a);(b);c_1+1,c_2,\ldots,c_A:(d);(e);(f))\right.$$

$$\left.- G_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f))\right].$$

The combination of (2.6.1) and (2.6.3) gives the differentiation of $G_n^{(u;h)}(x)$,

$$(2.6.4)\quad DG_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f))$$

$$= -nhx^{h-1}\frac{\Gamma((a)u+(b)n+(c)+(a)h)\ \Gamma((d)u+(f)n+(e))}{\Gamma((d)u+(f)n-(f)+(e)+(d)h)\ \Gamma((a)u+(b)n+(c))}$$

$$\times\ G_{n-1}^{(u+h;h)}(x:(a);(b);(b)+(c):(d);(e);(f)) .$$

Relation (2.6.3) leads to the relation

$$(1+\frac{a_1xD}{a_1u+b_1n+c_1})\ G_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f))$$

$$= G_n^{(u;h)}(x:(a);(b);c_1+1,c_2,\ldots,c_A:(d);(e);(f))$$

which on iteration yields

$$(2.6.5)\quad \prod_{r=1}^{A}(1+\frac{a_rxD}{a_ru+b_rn+c_r})\ G_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f))$$

$$= G_n^{(u;h)}(x:(a);(b);(c)+1:(d);(e);(f)) .$$

Rewriting (2.1.2) in the following form

$$G_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f)) = \frac{(-)^n\ \Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))}$$

$$\times\ x^{-u-e_1/d_1+1/d_1}\ \underset{u,h}{\Delta^n}\ [\frac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))}\ x^{u+e_1/d_1-1/d_1}]$$

and differentiating with respect to 'x', we obtain (after adjusting the parameters) the result

$$(2.6.6)\quad DG_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f))$$

$$= (-)^n \frac{(1-d_1u-e_1)\Gamma((d)u+(f)n+(e))}{d_1\,\Gamma((a)u+(b)n+(c))} x^{-u-1} \underset{u,h}{\Delta^n}\left[\frac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))} x^u\right]$$

$$+(-)^n \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))} x^{-u} \underset{u,h}{\Delta}\left[\frac{(d_1u+e_1-1)\Gamma((a)u+(b)n+(c))}{d_1\Gamma((d)u+(e))} x^{u-1}\right].$$

Hence

$$(2.6.7)\quad d_1xDG_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f)) = (d_1u+f_1n+e_1-1)$$

$$\times\, G_n^{(u-1;h)}(x:(a);(b);(a)+(c):(d);d_1+e_1-1,d_2+e_2,\ldots,d_D+e_D;(f))$$

$$-\,(d_1u+e_1-1)\,G_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f)).$$

From (2.6.6), we have

$$[\,(d_1u+e_1-1)+d_1xD\,]\,G_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f))$$

$$=(-)^n \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))} x^{-u} \underset{u,h}{\Delta^n}\left[\frac{\Gamma((a)u+(b)n+(c))\,x^u}{\Gamma(d_1u+e_1-1)\Gamma(d_2u+e_2)\ldots\Gamma(d_Du+e_D)}\right].$$

Repeating this process d_1 times, we get (d_2)

$$\prod_{j=1}^{d_1}[\,(d_1u+e_1-j)+d_1xD\,]\,G_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f))$$

$$= \frac{(-)^n \Gamma((d)u+(f)n+(e))\ x^{-u}}{\Gamma((a)u+(b)n+(c))} \underset{u,h}{\overset{n}{\triangle}} \left[\frac{\Gamma((a)u+(b)n+(c))\ x^u}{\Gamma(d_1u+e_1-d_1)\Gamma(d_2u+e_2)\dots\Gamma(d_Du+e_D)}\right].$$

which on iteration yields

$$(2.6.8)\quad \prod_{j_1=1}^{d_1} (d_1u+e_1-j_1+d_1xD)\dots\prod_{j_D=1}^{d_D} (d_Du+e_D-j_D+d_DxD)$$

$$\times \left[G_n^{(u;h)} (x : (a); (b); (c) : (d); (e); (f)) \right]$$

$$= (-)^n \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))} x^{-u} \underset{u,h}{\overset{n}{\triangle}} \left[\frac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e)-(d))} x^u \right]$$

$$= \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((d)u+(f)n+(e)-(d))} G_n^{(u-1;h)} (x: (a); (b); (a)+(c): (d); (e); (f)).$$

By putting $f(u) = (a_1u+b_1n+c_1)\dots(a_Au+b_An+c_A)$ in (2.5.4), we observe that

$$(2.6.9)\quad G_n^{(u;h)} (x: (a); (b); (c)+1 : (d); (e); (f))$$

$$= \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))} \sum_{r=0}^{\min(n,A)} (-)^r \binom{n}{r} \frac{\Gamma((a)u+(b)n+(c)+(a)hr)}{\Gamma((d)u+(f)n-(f)r+(e)+(d)hr)}$$

$$\times\ x^{hr}\, P_{A-r}\, G_{n-r}^{(u+hr;h)} (x: (a); (b); (c)+(b)r: (d); (e); (f)),$$

where $P_{A-r} = \underset{u,h}{\overset{r}{\triangle}} \left[(a_1u+b_1n+c_1)\dots(a_Au+b_An+c_A) \right]$

(Polynomial of degree $A-r$ in u).

Again if we let

$$f(u) = \prod_{j_1=1}^{d_1} (d_1u+e_1-j_1)\dots\prod_{j_D=1}^{d_D} (d_Du+e_D-j_D)$$

in (2.5.4), it gives

(2.6.10) $G_n^{(u-1;h)}(x:(a);(b);(a)+(c):(d);(e);(f))$

$$= \frac{\Gamma((d)u+(e)-(d)+(f)n)}{\Gamma((a)u+(b)n+(c))}\sum_{r=0}^{\min(n,k)}(-)^r\binom{n}{r}\frac{\Gamma((a)u+(b)n+(c)+(a)hr)}{\Gamma((d)u+(f)n-(f)r+(e)+(d)hr)}$$

$$\times\ x^{hr}\, Q_{k-r}\, G_{n-r}^{(u+hr;h)}(x:(a);(b);(c)+(b)r:(d);(e);(f)) ,$$

where $\underset{u,h}{\Delta^r}\left[\prod_{j_1=1}^{d_1}(d_1u+e_1-j_1)\dots\prod_{j_D=1}^{d_D}(d_Du+e_D-j_D)\right]$

$= Q_{k-r}$ (Polynomial of degree $k-r$ in u)

here, $k = d_1+\dots+d_D$.

Now equations (2.6.5) with (2.6.9) and (2.4.8) with (2.6.10), gives the following results

(2.6.11) $\prod_{r=1}^{A}(a_ru+b_rn+c_r+a_rxD)\ G_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f))$

$$= \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))}\sum_{r=0}^{\min(n,A)}(-)^r\binom{n}{r}\frac{\Gamma((a)u+(b)n+(c)+(a)hr)}{\Gamma((d)u+(f)n-(f)r+(e)+(d)hr)}$$

$$\times\ x^{hr}\ P_{A-r}\ G_{n-r}^{(u+hr;h)}\ (x : (a); (b); (c)+(b)r : (d); (e); (f))$$

and

$$(2.6.12)\quad \prod_{j_1=1}^{d_1} (d_1u+e_1-j_1+d_1xD) \ldots \prod_{j_D=1}^{d_D} (d_Du+e_D-j_D+d_DxD)$$

$$\times\ [\, G_n^{(u;h)}\ (x: (a); (b); (c) : (d); (e); (f))\,]$$

$$= \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))} \sum_{r=0}^{\min(n,k)} (-)^r \binom{n}{r} \frac{\Gamma((a)u+(b)n+(c)+(a)hr)}{\Gamma((d)u+(f)n-(f)r+(e)+(d)hr)}$$

$$\times\ x^{hr}\ Q_{k-r}\ G_{n-r}^{(u+hr;h)}\ (x: (a); (b); (c)+(b)r : (d); (e); (f)) .$$

Starting from (2.1.2) and using (2.5.3) for $n=1$, we obtain

$$\underset{u,h}{\Delta} [\, G_n^{(u;h)}\ (x : (a); (b); (c) : (d); (e); (f))\,]$$

$$= \underset{u,h}{\Delta} [(-)^n\, x^{-u} \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))}] \cdot \underset{u,h}{\Delta^n} [\frac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))} x^u]$$

$$+(-)^n\, x^{-u-h} \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c)+(d)h)} \underset{u,h}{\Delta^{n+1}} [\frac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))} x^u] ,$$

The use of the result $\underset{u,h}{\Delta} f(u) = f(u+h)-f(u)$, in first term of the R.H.S., gives us the following difference recurrence relation

$$(2.6.13)\quad \underset{u,h}{\Delta} [\, G_n^{(u;h)}\ (x : (a); (b); (c) : (d); (e); (f))\,]$$

$$= \left[\frac{\Gamma((d)u+(f)n+(e)+(d)h)\ \Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(f)n+(e))\ \Gamma((a)u+(b)n+(c)+(a)h)} x^{-h} - 1 \right]$$

$$\times\ G_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f)) - \frac{\Gamma((d)u+(f)n+(e)-(f)+(d)h)}{\Gamma((d)u+(f)n+(e))}$$

$$\times \frac{\Gamma((a)u+(b)n+(c))}{\Gamma((a)u+(b)n+(c)+(a)h)} x^{-h}\ G_{n+1}^{(u;h)}(x:(a);(b);(c)-(b)n:(d);(e);(f)).$$

We have

$$\Delta_{u,h}^{n+1} \left[\frac{\Gamma((a)u+(b)n+(b)+(c))}{\Gamma((d)u+(e))} x^u \right]$$

$$= \Delta_{u,h}^{n} \left[\frac{\Gamma((a)u+(b)n+(b)+(c)+(a)h)}{\Gamma((d)u+(e)+(d)h)} x^{u+h} \right] - \Delta_{u,h}^{n} \left[\frac{\Gamma((a)u+(b)n+(b)+(c))}{\Gamma((d)u+(e))} x^u \right].$$

In the above result the use of (2.1.2) gives still another recurrence relation

$$(2.6.14)\quad G_{n+1}^{(u;h)}(x:(a);(b);(c):(d);(e);(f)) = \frac{\Gamma((d)u+(f)n+(e)+(f))}{\Gamma((d)u+(f)n+(e))}$$

$$\times\ G_n^{(u;h)}(x:(a);(b);(b)+(c):(d);(e);(f)) - \frac{\Gamma((a)u+(b)n+(b)+(c)+(a)h)}{\Gamma((a)u+(b)n+(b)+(c))}$$

$$\times \frac{\Gamma((d)u+(f)n+(e)+(d)h)}{\Gamma((d)u+(f)n+(f)+(e))} x^h\ G_n^{(u+h;h)}(x:(a);(b);(b)+(c):(d);(e);(f)).$$

Rewrite (2.6.4) as

$$(x^{1-h}D)\ G_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f))$$

$$= \frac{-nh\ \Gamma((a)u+(b)n+(c)+(a)h)\ \Gamma((d)u+(f)n+(e))}{\Gamma((d)u+(f)n-(f)+(e)+(d)h)\ \Gamma((a)u+(b)n+(c))}$$

$$X\ G_{n-1}^{(u+h;h)}(x:(a);(b);(b)+(c):(d);(e);(f))\ .$$

clearly by operating $(x^{1-h}D)$, s times, we get

$$(2.6.15)\quad (x^{1-h}D)^s\ G_n^{(u;h)}(x:(a);(b);(c)\ :(d);(e);(f))$$

$$= \frac{(-n)_s\ h^s\ \Gamma((a)u+(a)hs+(b)n+(c))\ \Gamma((d)u+(f)n+(e))}{\Gamma((d)u+(f)n-(f)s+(e)+(d)hs)\ \Gamma((a)u+(b)n+(c))}$$

$$X\ G_{n-s}^{(u+hs;h)}(x\ :(a);(b);(b)s+(c)\ :(d);(e);(f))\ .$$

Next, taking $f(u) = \dfrac{\Gamma((a)u+(b)n+(c))\ x^u}{\Gamma((d)u+(e))\ (-u/h)_n}$ in the result [79]

$$(-)^n \underset{u,h}{\Delta^n} (-u/h)_n\ f(u) = (E_{u,h}\ u/h-u/h)_n\ f(u)\ ,$$

we get

$$(-)^n \underset{n,h}{\Delta^n} \left[\frac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))}\ x^u \right]$$

$$= (E_{u,h}\ u/h-u/h)_n \left[\frac{\Gamma(u/h-n+1)\ \Gamma((a)u+(b)n+(c))}{\Gamma(u/h+1)\ \Gamma((d)u+(e))}\ x^u \right]$$

Hence

$$(2.6.16)\quad G_n^{(u;h)}(x : (a);(b);(c) : (d);(e);(f))$$

$$= \frac{(-)^n \Gamma((d)u+(f)n+(e))}{\Gamma(u/h+1)\ \Gamma((a)u+(b)n+(c))} (x^h E_{u,h} - u/h)_n \left[\frac{\Gamma(u/h-n+1)\ \Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))}\right]$$

By induction method, we can easily show that

$$(2.6.17)\quad (u,h)_n \underset{u,h}{\overset{n}{\Delta}} [f(u)] = \prod_{j=1}^{n} [u \underset{u,h}{\Delta} -nh+jh]\ f(u)\ ,$$

where $(u,h)_n = u(u+1)\ldots(u+(n-1)h)$.

Substituting $f(u) = \dfrac{\Gamma((a)u+(b)n+(c)}{\Gamma((d)u+(e))} x^u$ and simplifying it,

we get the operational formula

$$(2.6.18)\quad G_n^{(u;h)}(x : (a);(b);(c) : (d);(e);(f))$$

$$= \frac{(-)^n \Gamma((a)u+(b)n+(c))\ x^{-u}}{(u,h)_n \Gamma((d)u+(f)n+(e))} \prod_{j=1}^{n} [u \underset{u,h}{\Delta} -nh+jh]\left[\frac{\Gamma((a)u+(b)n+(c))\ x^u}{\Gamma((d)u+(e))}\right]$$

$$= \frac{(-)^n \Gamma((d)u+(f)n+(e))}{(u,h)_n \Gamma((d)u+(b)n+(c))} \prod_{j=1}^{n} [u(x^h E_{u,h}-1)-nh+jh]\ \frac{\Gamma((a)u+(b)n+(c))}{\Gamma((d)u+(e))} .$$

2.7 Some Finite Expansions : Putting $g(u) = \Gamma((a)u+(c))\, x^u / \Gamma((d)u+(e))$ in the well known result

$$(2.7.1) \quad g(u+hk) = \sum_{r=0}^{k} \binom{k}{r} \Delta_{u,h}^{r}\, g(u)$$

and using (2.1.2), we get the expansion of x^{hk} in terms of $G_r^{(u;h)}(x)$,

$$(2.7.2) \quad x^{hk} = \frac{\Gamma((d)u+(e)+(d)hk)\ \Gamma((a)u+(c))}{\Gamma((a)u+(c)+(a)hk)}$$

$$\times \sum_{r=0}^{k} \binom{k}{r} \frac{(-)^r}{\Gamma((d)u+(e)+(f)r)}\, G_r^{(u;h)}(x:(a);(b);(c)-(b)r:(d);(e);(f)).$$

Next substituting $g(u) = G_n^{(u-hk;h)}(x:(a);(b);(c):(d);(e);(f))$ in (2.7.1) and taking the help of (2.2.1), we get another expansion formula

$$(2.7.3) \quad G_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f)) = \sum_{r=0}^{k} \sum_{m=0}^{r} (-)^{r-m}$$

$$\times \binom{k}{r}\binom{r}{m} G_n^{(u+hm-hk;h)}(x:(a);(b);(c):(d);(e);(f)) .$$

Further, if we take $h = 1$ and $f(u) = \dfrac{\Gamma((d)u+(e)\ y^u}{\Gamma((a)u+(b)n+(c))}$ in (2.5.4), we obtain the following expansion of $(1-xy)^n$

$$(2.7.4)\quad (1-xy)^n = \sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma((a)u+(b)n+(c)+(a)hr)}{\Gamma((a)u+(b)n+(c)+(g)r)}$$

$$\times \frac{\Gamma((d)u+(e))\ x^r}{\Gamma((d)u+(f)n-(f)r+(e)+(d)r)}\ G_{n-r}^{(u+r;1)}(x:(a);(b);(c)+(b)r:(d);(e);(f))$$

$$\times\ G_r^{(u;1)}(y:(d);(0);(e):(a);(b)n+(c);(g)).$$

Now consider

$$\underset{u,h}{\Delta^n}\left[\frac{\Gamma((a)u+(b)n+(c))\ \Gamma((a')u+(b')n+(c'))}{\Gamma((d)u+(e))\ \Gamma((d')u+(e'))}(xy)^u\right.$$

$$=\sum_{r=0}^{n}\binom{n}{r}\underset{u,h}{\Delta^{n-r}}\left[\frac{\Gamma((a)u+(b)n+(c)+(a)hr)}{\Gamma((d)u+(e)+(d)hr)}x^{u+hr}\right]$$

$$\times\underset{u,h}{\Delta^{r}}\left[\frac{\Gamma((a')u+(b')n+(c'))}{\Gamma((d')u+(e'))}y^u\right]$$

and apply (2.1.2) to get still another expansion formula

$$(2.7.5)\quad G_n^{(u;h)}(xy:(a),(a');(b),(b');(c),(c'):(d),(d');(e),(e');(f),(f'))$$

$$=\frac{\Gamma((d)u+(f)n+(e)\,\Gamma((d')u+(f')n+(e'))}{\Gamma((a)u+(b)n+(c))}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma((a)u+(b)n+(c)+(a)hr)}{\Gamma((d')u+(f')r+(e'))}$$

$$\times\frac{x^{hr}}{\Gamma((d)u+(e)+(f)n-(f)r+(d)hr)}\ G_{n-r}^{(u+hr;h)}(x:(a);(b);$$

$$(c)+(b)r:(d);(e);(f))\ G_r^{(u;h)}(y:(a');(b');(c')+(b')n-(b')r:(d');(e');(f')) .$$

It is easy to see that

$$\Delta_{u,h}^{m+n}\left[\frac{\Gamma((a)u+(b)m+(b)n+(c))}{\Gamma((d)u+(e))}x^u\right]$$

$$=\sum_{r=0}^{m}(-)^{m-r}\binom{m}{r}\Delta_{u,h}^{n}\left[\frac{\Gamma((a)u+(b)m+(b)n+(c)+(a)hr)}{\Gamma((d)u+(e)+(d)hr)}x^{u+hr}\right]$$

from which it can easily be shown that

$$(2.7.6)\quad G_{m+n}^{(u;h)}(x:(a);(b);(c):(d);(e);(f))$$

$$=\frac{\Gamma((d)u+(f)m+(f)n+(e))}{\Gamma((a)u+(b)m+(b)n+(c))}\sum_{r=0}^{m}(-)^{r}\binom{m}{r}\frac{\Gamma((a)u+(b)m+(b)n+(c)+(a)hr)}{\Gamma((d)u+(f)n+(e)+(d)hr)}$$

$$\times\ x^{hr}\ G_n^{(u+hr;h)}(x:(a);(b);(c)+(b)m:(d);(e);(f)) .$$

Next we know that (see [79])

$$\Delta_{u,mh}^{n}f(u)=\sum_{r_1,\ldots,r_n}^{m-1}\Delta_{u,h}^{n}f(u+Rh)$$

and

$$\Delta_{u,2h}^{n}f(u)=\sum_{r=0}^{n}\binom{m}{r}\Delta_{u,h}^{n}f(u+rh) ,$$

where $R=r_1+\ldots+r_m$.

Bu substituting $f(u) = \Gamma((a)u+(b)n+(c))x^u / \Gamma((d)u+(e))$ in the above results, it follows that

$$(2.7.7) \quad G_n^{(u;mh)}(x:(a);(b);(c):(d);(e);(f))$$

$$= \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))} \sum_{r_1,\ldots,r_n=0}^{m-1} \frac{\Gamma((a)u+(b)n+(c)+(a)hR)}{\Gamma((d)u+(f)n+(e)+(d)hR)} x^{hR}$$

$$\times \; G_n^{(u+Rh;h)}(x:(a);(b);(c):(d);(e);(f))$$

and

$$(2.7.8) \quad G_n^{(u;2h)}(x:(a);(b);(c):(d);(e);(f))$$

$$= \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))} \sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma((a)u+(b)n+(c)+(a)hr)}{\Gamma((d)u+(f)n+(e)+(d)hr)} x^{hr}$$

$$\times \; G_n^{(u+hr;h)}(x:(a);(b);(c):(d);(e);(f)).$$

If we make the substitution $f(u) = \dfrac{\Gamma((a)u+(b)mn+(c))}{\Gamma((d)u+(e))} x^u$ in another known result

$$\Delta_{u,h}^{mn} f(u) = \sum_{k=0}^{n} (-)^{n-k} \binom{n}{k} \Delta_{u,h}^{(m-1)n} f(u+kh) \;,$$

we obtain

$$(2.7.9) \quad G_{mn}^{(u;h)}(x:(a);(b);(c):(d);(e);(f))$$

$$= \frac{\Gamma((d)u+(f)mn+(e))}{\Gamma((a)u+(b)mn+(c))} \sum_{k=0}^{n} (-)^k \binom{n}{k} \frac{\Gamma((a)u+(b)mn+(c)+(a)hk)}{\Gamma((d)u+(f)mn-(f)n+(e)+(d)hk)}$$

$$\times\ x^{hk}\ G^{(u+hk;h)}_{(m-1)n}\ (x : (a); (b); (c)+(b)n : (d); (e); (f)) \ .$$

Repeated application of (2.7.9) gives

$$(2.7.10)\quad G^{(u;h)}_{mn}\ (x: (a); (b); (c): (d); (e); (f)) = \frac{\Gamma((d)u+(f)mn+(e))}{\Gamma((a)u+(b)mn+(c))}$$

$$\times \sum_{k_1,\ldots,k_{m-1}=0}^{n} (-)^k \binom{n}{k_1}\cdots\binom{n}{k_{m-1}} \frac{\Gamma((a)u+(b)mn+(c)+(a)hK)}{\Gamma((d)u+(f)n+(e)+(d)hK)}$$

$$\times\ G^{(u+hk;h)}_{n}\ (x: (a); (b); (c)+(b)mn-(b)n: (d); (e); (f)) \ ,$$

where $K = k_1+\ldots+k_{m-1}$.

Further, the explicit form (2.2.2) of $G^{(u;h)}_{n}\ (x)$ has the following alternative form

$$(2.7.11)\quad G^{(u;h)}_{n}\ (x: (a); (b); (c): (d); (e); (f)) = \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))}\ x^{-e_1/d_1}$$

$$\times \sum_{r=0}^{n} (-)^r \binom{n}{r}\ E^{r}_{(e),(d)h} \left[\frac{x^{e_1/d_1}}{\Gamma((d)u+(e))}\right] E^{n-r}_{(c),-(a)h} \left[\Gamma((a)u+(b)n+(c)+(a)hn)\right]$$

where $E_{(e),(d)h} \equiv E_{e_1,d_1h} \ldots E_{e_D,d_Dh}$

and $E_{(c),-(a)h} \equiv E_{c_1,-a_1h} \ldots E_{c_A,-a_Ah}$,

or

$$(2.7.12)\quad G_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f)) = \frac{\Gamma((d)u+(f)n+(e))}{\Gamma((a)u+(b)n+(c))}\, x^{-e_1/d_1}$$

$$\times\left(\Delta_{(c),-(a)h} - \Delta_{(e),(d)h}\right)^n \left[\frac{\Gamma((a)u+(b)n+(c)+(a)hn)}{\Gamma((d)u+(e))}\, x^{-e_1/d_1}\right]$$

where $\Delta_{(c),-(a)h} = E_{(c),-(a)h} - 1$ and $\Delta_{(e),(d)h} = E_{(e),(d)h} - 1$.

Lastly, if $a_1h, \ldots, a_Ah, d_1h, \ldots, d_Dh$ all are positive integers from (2.7.12) above, we can easily obtain the following expansion

$$(2.7.13)\quad G_n^{(u;h)}(x:(a);(b);(c):(d);(e);(f))$$

$$= \frac{\Gamma((d)u+(f)n+(e))\;\Gamma((a)u+(b)n+(c)+(a)hn)}{\Gamma((d)u+(e))\;\Gamma((a)u+(b)n+(c))} \sum_{r=0}^{n} (-)^{n-r}\binom{n}{r}$$

$$\times\ {}_1F_{(d_1+\dots+d_D)h}\left[\begin{matrix}-r & ; \\ \Delta((d)h;(d)u+(e)); & \end{matrix} \underset{d_1\ \ldots d_D\cdot h}{\left(\frac{x}{d_1\ \ d_D\ (d_1+\dots+d_D)}\right)^h}\right]$$

$$\times\ {}_1F_{(a_1+\dots+a_A)h}\left[\begin{matrix}-n+r & ; \\ \Delta((a)h;(1-(a)u-(b)n-(c)-(a)hn); & \end{matrix} \underset{a_1\ \ldots a_A\cdot h}{\left(\frac{1}{a_1\ \ a_A\ (a_1+\dots+a_A)}\right)^h}\right]$$

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CHAPTER III

RODRIGUES' TYPE FORMULA FOR GENERALIZED HYPERGEOMETRIC FUNCTIONS OF TWO VARIABLES AND ITS APPLICATIONS

3.1 Introduction : Various authors have made successful efforts for finding out the new summation formulas and transformations for Kampe' de Fe'riet double hypergeometric series. In recent past Carlitz [26,27,28,29,30], Jain [55], Pandey and Saran [77], Srivastava and Saran [108, 109, 110], Srivastava [107], Shah [88], Singal [98, 99, 100, 101, 102, 103], Sharma and Abiodun [91] and Sharma [90] gave a number of summation formulas and transformations of $F^{(2)}[z,1]$ (or $F^{(2)}[1,z]$) and $F^{(2)}[1,1]$ defined by (1.3.9). Most of the result are obtained purely by series manipulation techniques.

The aim of the present Chapter is to give a Rodrigues' type representations for Kampé de Fériet double hypergeometric series and make use of these representations to obtain some more general type of summation formulas and transformations for $F^{(2)}[z,1]$ and $F^{(2)}[1,1]$. Some of the summation formulas and transformations thus obtained are believed to be new.

3.2 Rodrigues' Type Formula : In view of (1.5.5), we have

$$\Delta_u^n \left[\frac{\Gamma((a)+u)}{\Gamma((b)+u)} \frac{\Gamma((e)+u)}{\Gamma((f)+u)} Y^u \; {}_{A+C}F_{B+D}\left[\begin{matrix}(a)+u\ , & (c)\ ; \\ (b)+u\ , & (d)\ ;\end{matrix}\; x\right] \right]$$

$$= \sum_{r=0}^{n} (-)^{n-r} \binom{n}{r} \frac{\Gamma((a)+u+r)\,\Gamma((e)+u+r)}{\Gamma((b)+u+r)\,\Gamma((f)+u+r)} y^{u+r} \sum_{s=0}^{\infty} \frac{((a)+u+r)_s ((c))_s}{((b)+u+r)_s ((d))_s} \frac{x^s}{s!},$$

$$= \frac{(-)^n \Gamma((a)+u)}{\Gamma((b)+u)} \frac{\Gamma((e)+u)\, y^u}{\Gamma((f)+u)}$$

$$\times \sum_{r=0}^{n} \sum_{s=0}^{\infty} \frac{((a)+u)_{r+s}}{((b)+u)_{r+s}} \frac{(-n)_r}{((f)+u)_r} \frac{((e)+u)_r}{((d))_s} \frac{((c))_s}{r!\ s!} x^s y^r,$$

Therefore the Rodrigues' type of formula for $F^{(2)}[x,y]$ can be given by

$$(3.2.1)\quad F\left[\begin{matrix}(a)+u:-n,\ (e)+u;\ (c);\\ (b)+u:\qquad (f)+u;\ (d);\end{matrix}\ y,x\right] = \frac{(-)^n \Gamma((a)+u)\ \Gamma((f)+u)\, y^{-u}}{\Gamma((a)+u)\qquad \Gamma((e)+u)}$$

$$\times\ \Delta_u^n \left[\frac{\Gamma((a)+u)\,\Gamma((e)+u)\, y^u}{\Gamma((b)+u)\,\Gamma((f)+u)}\ {}_{A+C}F_{B+D}\left[\begin{matrix}(a)+u,\ (c)\ ;\\ (b)+u,\ (d)\ ;\end{matrix}\ x\right]\right].$$

Next consider

$$\Delta_u^n.\left[\frac{\Gamma((a)+u)\,\Gamma((\cancel{c})+u)}{\Gamma((b)+u)\,\Gamma((f)+u)}\, y^u\ {}_{1+A+C}F_{B+D}\left[\begin{matrix}-n+u,\ (a)+u,\ (c)\ ;\\ (b)+u,\ (d)\ ;\end{matrix}\ x\right]\right]\quad e/$$

$$= \frac{(-)^n \Gamma((a)+u)\,\Gamma((e)+u)\, y^u}{\Gamma((b)+u)\ \Gamma((f)+u)} \sum_{r,s\geq 0} \frac{(-n)_r ((a)+u)_r ((e)+u)_r}{((b)+u)_r\ ((f)+u)_r}$$

$$\times\ \frac{(-n+u+r)_s\ ((a)+u+r)_s\ ((c))_s}{((b)+u+r)_s\ ((d))_s} x^s y^r.$$

Hence, we get

$$(3.2.2)\quad \frac{\Gamma((a))\Gamma((e))}{\Gamma((b))\Gamma((f))}(-)^n\, F\left[\begin{matrix}-n,(a) & : & (c) & ; & (e) & ; \\ & & & & & & x,y \\ (b) & : & (d) & ; & (f) & ;\end{matrix}\right]$$

$$= \lim_{u\to 0}\left[\Delta_u^n \frac{\Gamma((a)+u)\Gamma((e)+u)}{\Gamma((b)+u)\Gamma((f)+u)}\, y^u\; {}_{1+A+C}F_{B+D}\left[\begin{matrix}-n+u,(a)+u,(c); \\ & x \\ (b)+u,(d);\end{matrix}\right]\right].$$

3.3 <u>Theorem I</u> : If

$$(3.3.1)\quad {}_{A+B}F_{F+G}\left[\begin{matrix}(a)+u,(b)\ ; \\ & x \\ (f)+u,(g)\ ;\end{matrix}\right] = \frac{y\,\Gamma((d)+u)}{\Gamma((k)+u)},$$

where y is some function of x but independent of u, then

$$(3.3.2)\quad F\left[\begin{matrix}(a) & : & (b) & ; & -n,(c) & ; \\ & & & & & & \cancel{x,y} \\ (f) & : & (g) & ; & (h) & ;\end{matrix}\right] \qquad z,\ x$$

$$= \frac{y\,\Gamma((d))}{\Gamma((k))}\; {}_{1+A+C+D}F_{F+H+K}\left[\begin{matrix}-n,(a),(c),(d)\ ; \\ & z \\ (f),(h),(k)\ ;\end{matrix}\right].$$

<u>Proof</u> : From (3.3.1), we can write

$$\Delta_u^n\left[\frac{\Gamma((a)+u)\Gamma((c)+u)}{\Gamma((f)+u)\Gamma((h)+u)}\, z^u\; {}_{A+B}F_{F+G}\left[\begin{matrix}(a)+u, & (b)\ ; \\ & & x \\ (f)+u, & (g)\ ;\end{matrix}\right]\right]$$

$$= \Delta_u^n\left[\frac{\Gamma((a)+u)\Gamma((c)+u)\Gamma((d)+u)}{\Gamma((f)+u)\Gamma((h)+u)\Gamma((k)+u)}\, yz^u\right].$$

Using the Rodrigues' type formulae (3.2.1) and (1.6.6), we get the required result (3.3.2). Thus the theorem is proved.

<u>Special Cases of Theorem I</u> : Here we shall discuss some of the possible special cases of the above theorem.

(i) In Saalchütz theorem [85, p.87] replacing a by a+u and c by f+u, we get

$$(3.3.3)\quad {}_3F_2\left[\begin{matrix}-m, a+u, b & ; \\ f+u, 1+a+b-f-m & ;\end{matrix}\ 1\right] = \frac{(f-a)_m\ (f-b+u)_m}{(f+u)_m\ (f-a-b)_m}\ .$$

Applying theorem I, we obtain the transformation

$$(3.3.4)\quad F\left[\begin{matrix}a: -m, b & ; & -n, (c) & ; \\ f: 1+a+b-f-m & ; & (h) & ;\end{matrix}\ 1, z\right]$$

$$= \frac{(f-a)_m\ (f-b)_m}{(f)_m\ (f-a-b)_m}\ {}_{C+3}F_{H+2}\left[\begin{matrix}-n, a, f-b+m, (c) & ; \\ f-b, f+m, \quad (h) & ;\end{matrix}\ z\right],$$

or alternatively, we can write

$$(3.3.5)\quad F\left[\begin{matrix}a:\ -m,\ b & ; & -n, f-b, f+m, (c) & ; \\ f: 1+a+b-f-m & ; & a, f-b+m, \quad (h) & ;\end{matrix}\ 1, z\right]$$

$$= \frac{(f-a)_m\ (f-b)_m}{(f)_m\ (f-a-b)_m}\ {}_{C+1}F_H\left[\begin{matrix}-n,\ (c) & ; \\ (h) & ;\end{matrix}\ z\right].$$

The above transformation is very general in nature and is believed to be new. It contains several known result, a few of them will be discussed here.

Also for 1+a-f-m=g (from (3.3.4)), we have

$$(3.3.6)\quad F\left[\begin{matrix}a\ : & -m, f+g+m-a-1 & ; -n, & (c) & ; \\ f\ : & g & ; & (h) & ;\end{matrix}\ 1, z\right]$$

$$= \frac{(f-a)_m\ (g-a)_m}{(f)_m\ (g)_m}\ {}_{C+3}F_{H+2}\left[\begin{matrix} -n, a, 1+a-g, (c) & ; \\ 1+a-g-m, f+m, (h) & ; \end{matrix}\ z\right],$$

which is the generalized form of a transformation established earlier by Srivastava and Saran [110, (2.1)].

Further, if we set $H=1$, $C=1$, $h_1=a$, $c_1=f+n-1$ and $z=1$ in (3.3.6), we get the summation formula

$$(3.3.7)\quad F\left[\begin{matrix} a: & -m, f+g+m-a-1 ; & -n, f+n-1 ; \\ f: & g ; & a ; \end{matrix}\ 1,1\right]$$

$$= \frac{m!\,(f-a)_m\ (g-a)_{m-n}\ (f+g-a+m-1)_n}{(m-n)!\ (f)_{m+n}\ (g)_m}\ .$$

The substitution $f=a+c$ and $g=a$ in the above result leads to another formula of Srivastava and Saran [108, p.437] (in its correct form, as the result given in their paper is erroneous)

$$F\left[\begin{matrix} a: & -m, a+c+m-1 ; & -n, a+c+n-1 ; \\ a+c: & a ; & a ; \end{matrix}\ 1,1\right]$$

$$= 0,\ \text{when } m \neq n$$

$$= \frac{n!\,\Gamma(a+c)}{(a+c+2n-1)\,\Gamma(a+c+n-1)} \cdot \frac{(c)_n}{(a)_n}\ ;\ \text{when } m=n.$$

Further, the substitution $C=H=1$, $c_1=c$, $h_1=h$, $f+g=1+a+b-m$, $f+h=1+a+c-n$ and $z=1$ in (3.3.6), leads to another transformation

$$(3.3.8)\quad F\left[\begin{matrix} a : -m, b ; -n, c ; \\ f : \quad g ; \quad h ; \end{matrix}\ 1,1\right]$$

$$= \frac{(f-a)_m\,(g-a)_m}{(f)_m\,(g)_m}\ {}_4F_3\left[\begin{matrix} -n, a, f-b+m, c ; \\ f+m, f-b, \ h ; \end{matrix}\ 1\right],$$

which is due to Srivastava and Saran [109, (3.1)].

Again if we take $C=H=6$, $c_1=g$, $c_2=1+g/2$, $c_3=h$, $c_4=c$, $c_5=d$, $c_6=1+2g-h-c-d+n$, $h_1=g/2$, $h_2=1+g-h$, $h_3=1+g-c$, $h_4=1+g-d$, $h_5=c+d+h-g-n$, $h_6=1+g+n$ and $z=1$ in (3.3.5), we get

$$(3.3.9)\quad F\left[\begin{matrix} a : \ -m, b \ ; -n, f-b, f+m, g, 1+g/2, h, c, d, 1+2g-h-c-d+n ; \\ f : 1+a+b-f-m ; a, f-b+m, g/2, 1+g-h, 1+g-c, 1+g-d, c+d+h-g-n, 1+g+n ; \end{matrix}\ 1,1\right] = \frac{(f-a)_m\,(f-b)_m}{(f)_m\,(f-a-b)_m}\times$$

$$\times\ {}_7F_6\left[\begin{matrix} -n, g, 1+g/2, c, d, h, 1+2g-h-c-d+n ; \\ g/2, 1+g-c, 1+g-d, 1+g-h, c+d+h-g-n, 1+g+n ; \end{matrix}\ 1\right].$$

If we apply the well known Dougall's theorem [104, (2.3.4.4), p.56] in terminating form, we get the summation formula

$$(3.3.10)\quad F\left[\begin{matrix} a: -m, b; -n, f-b, f+m, g, 1+g/2, c, d, h, 1+2g-h-c-d+n ; \\ f: 1+a+b-f-m; a, f-b+m, g/2, 1+g-h, 1+g-c, 1+g-d, c+d+h-g-n, 1+g+n ; \end{matrix}\ 1,1\right]$$

$$= \frac{(f-a)_m \ (f-b)_m \ (1+g)_n \ (1+g-h-c)_n \ (1+g-h-d)_n \ (1+g-c-d)_n}{(f)_m \ (f-a-b)_m \ (1+g-c)_n \ (1+g-d)_n \ (1+g-h)_n \ (1+g-h-c-d)_n} ,$$

which is believed to be new.

Again in (3.3.4) the substitution $C=H=1$, $c_1=c$, $h_1=1+c-h-n$ and $z=1$ leads to

$$(3.3.11) \quad F\left[\begin{matrix} a : -m,\ b \quad ; -n,\ c \quad ; \\ f : 1+a+b-f-m \ ; 1+c-h-n \ ; \end{matrix} \ 1,1\right]$$

$$= \frac{(f-a)_m \ (f-b)_m}{(f)_m \ (f-a-b)_m} \ {}_4F_3\left[\begin{matrix} -n,\ a,\ c,\ f-b+m \ ; \\ f-b,\ f+m,\ 1+c-h-n \ ; \end{matrix} \ 1\right].$$

Applying the following transformation for double hypergeometric series by Singal [101, (2.2)]

$$(3.3.12) \quad F\left[\begin{matrix} b : -n\ ,\ c\ ;\ -m\ ,\ c'\ ; \\ d : \quad e\ ;\ 1+c'-e'-m\ ; \end{matrix} \ 1,1\right]$$

$$= \frac{(e')_m}{(e'-c')_m} \ F\left[\begin{matrix} - : -n,b,c\ ;\ -m,d-b,c'\ ; \\ d : \quad e\ ; \quad e'\ ; \end{matrix} \ 1,1\right],$$

to the R.H.S. it becomes

$$= \frac{(h)_n}{(h-c)_n} \ F\left[\begin{matrix} - : -m,a,b \quad ;\ -n,f-a,c\ ; \\ f : 1+a+b-f-m\ ; \quad h\ ; \end{matrix} \ 1,1\right].$$

Therefore

$$(3.3.13)\quad F\left[\begin{array}{l} - : -n, a, b \; ; \; -m, f-a, c \; ; \\ f : 1+a+b-f-m \; ; \quad h \; ; \end{array} 1,1\right]$$

$$= \frac{(f-a)_m \, (f-b)_m \, (h-c)_n}{(f)_m \, (h-c)_n \, (f-a-b)_m} \; {}_4F_3\left[\begin{array}{l} -n, a, c, f-b+m \; ; \\ f+m, f-b, 1+c-h-n \; ; \end{array} 1,1\right]$$

which is one of the two main transformations established earlier by Singal [98]. He has also discussed its various possible special cases.

(ii) In Gauss's theorem [85, p.49]

$$(3.3.14)\quad F(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \; ; \text{ for } \mathrm{Re}\,(c-a-b) > 0 ,$$

replacing a by a+u, c by c+u and applying theorem I we get still another transformation

$$(3.3.15)\quad F\left[\begin{array}{l} a : b \; ; \; -n, (d) \; ; \\ c : - \; ; \quad (h) \; ; \end{array} 1,z\right]$$

$$= \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \; {}_{D+2}F_{H+1}\left[\begin{array}{l} -n, \; a, \; (d) \; ; \\ c-b \; , \; (h) \; ; \end{array} z\right].$$

The above formula for D=H=0 and z=1, reduces to the terminating form of the well known result [15, p.22]

$$F_1[a:b,-n;c;1,1] = \frac{\Gamma(c)\,\Gamma(c-a-b+n)}{\Gamma(c-a)\,\Gamma(c-b+n)} \; ; \text{ for } \mathrm{Re}\,(c-a-b+n) > 0.$$

From (3.3.15) it is also follows that (take $D=H=1$, $d_1 = d$, $h_1 = 1+a+b+d-c-n$ and $z = 1$)

$$(3.3.16)\quad F\left[\begin{matrix} a : b; & -n, d ; \\ c : -; & 1+a+b+d-c-n ; \end{matrix}\; 1,1\right] = \frac{\Gamma(c)\Gamma(c-a-b)(c-a-b)_n(c-b-d)_n}{\Gamma(c-a)\Gamma(c-b)(c-b)_n(c-a-b-d)_n} .$$

(iii) In the formula (3.3.14) replace c by $c+u$ and ultimately use the theorem I, to get the transformation

$$(3.3.17)\quad F\left[\begin{matrix} - : & a, b ; & -n, (d) ; \\ c : & - ; & (h) ; \end{matrix}\; 1,z\right]$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\;\Gamma(c-b)}\; {}_{D+2}F_{H+2}\left[\begin{matrix} -n, c-a-b, (d) ; \\ c-a, c-b, (h) ; \end{matrix}\; z\right].$$

In case we put $D=d_1=c-b$, $H=0$ and $z=1$ in (3.3.17), we get the following summation formula for terminating F_3

$$(3.3.18)\quad F_3[-n, a; b, c-b; c; 1,1] = \frac{\Gamma(c)\Gamma(c-a-b)\;(b)_n}{\Gamma(c)\Gamma(c-b)\;(c-a)_n} .$$

Lastly, if we set $D=2$, $H=1$, $d_1=c-b$, $d_2=d$, $h_1=1+d-b-n$ in (3.3.17), we get still another summation formula

$$(3.3.19)\quad F\left[\begin{matrix} -: a, b; & -n, c-b, d ; \\ c: \quad -; & 1+d-b-n ; \end{matrix}\; 1,1\right] = \frac{\Gamma(c)\Gamma(c-a-b)(b)_n\;(c-a-d)_n}{\Gamma(c-a)\Gamma(c-b)(c-a)_n\;(c-d)_n} .$$

3.4. Theorem II : If

$$(3.4.1) \quad {}_{A+B+1}F_{F+G}\left[\begin{matrix}-n+u,(a)+u,(b);\\(f)+u,(g);\end{matrix}\ x\right] = \frac{y((d)+u)_{n-u}\ ((e))_{n-u}}{((j)+u)_{n-u}\ ((k))_{n-u}} ,$$

where y is some function of x and is independent of u, then

$$(3.4.2) \quad F\left[\begin{matrix}-n,(a):(b);(c);\\(f):(g);(h);\end{matrix}\ x,y\right] = \frac{y((d))_{n}\ ((e))_{n}}{((j))_{n}\ ((k))_{n}}\ X$$

$$X\ {}_{A+C+J+K+1}F_{D+E+F+H}\left[\begin{matrix}-n,(a),(c),(j),1-n-(k)\ ;\\(f),(h),(d),1-n-(e)\ ;\end{matrix}\ z\right].$$

Proof : From equation (3.4.1), we have

$$\Delta_u^n\left[\frac{\Gamma((a)+u)\Gamma((c)+u)}{\Gamma((f)+u)\Gamma((h)+u)}\ z^u\ {}_{A+B+1}F_{F+G}\left[\begin{matrix}-n+u,(a)+u,(b)\ ;\\(f)+u,(g)\ ;\end{matrix}\ x\right]\right]$$

$$= \Delta_u^n\left[\frac{((d)+u)_{n-u}\ ((e))_{n-u}\ \Gamma((a)+u)\Gamma((c)+u)}{((j)+u)_{n-u}\ ((k))_{n-u}\ \Gamma((f)+u)\Gamma((h)+u)}\ z^u y\right]$$

$$= y\sum_{r=0}^{n}\frac{(-)^n\ (-n)_r\ ((d)+u+r)_{n-u-r}\ ((e))_{n-u-r}}{r!\ ((j)+u+r)_{n-u-r}\ ((k))_{n-u-r}}$$

$$X\ \frac{\Gamma((a)+u+r)\Gamma((c)+u+r)}{\Gamma((f)+u+r)\Gamma((h)+u+r)}\ z^u ,$$

which after taking Lim $u \rightarrow 0$ and using Rodriques' type formula (3.2.2), gives the required transformation (3.4.2).

<u>Special Cases of theorem II</u> : (i) In the Saalschütz theorem [85, p.87] replacing n by n-u, c by f+u and adjusting the parameters, we obtain

$$(3.4.3) \quad {}_3F_2\left[\begin{matrix}-n+u, & a, b & ; \\ f+u, & 1+a+b-f-n & ;\end{matrix}\; 1\right] = \frac{(f-a+u)_{n-u}\;(f-b+u)_{n-u}}{(f+u)_{n-u}\;(f-a-b+u)_{n-u}} .$$

Because of its resemblence with (3.4.1), of theorem II, we get by (3.4.2), the transformation

$$(3.4.4) \quad F\left[\begin{matrix}-n : & a,\ b & ; & (c) & ; \\ f : & 1+a+b-f-n & ; & (h) & ;\end{matrix}\; 1,z\right]$$

$$= \frac{(f-a)_n\;(f-b)_n}{(f)_n\;(f-a-b)_n}\; {}_{C+2}F_{H+2}\left[\begin{matrix}-n, & f-a-b, & (c) & ; \\ f-a, & f-b\ , & (h) & ;\end{matrix}\; z\right],$$

or alternatively we can write

$$(3.4.5) \quad F\left[\begin{matrix}-n : & a,b & ; & (c) & ; \\ f : & g & ; & (h) & ;\end{matrix}\; 1,z\right]$$

$$= \frac{(f-a)_n\;(g-a)_n}{(f)_n\;(g)_n}\; {}_{C+2}F_{H+2}\left[\begin{matrix}-n, & f-a-b, & (c) & ; \\ f-a, & f-b, & (h) & ;\end{matrix}\; z\right],$$

provided, $f+g = 1+a+b-n$.

By specializing the parameters in (3.4.4) or in (3.4.5), a number of transformations and summation formulas can be obtained. For example, if $C=2$, $H=1$, $c_1=c$, $c_2=d$, $h_1=h$ and $z=1$ in (3.4.4), we get the following result

$$(3.4.6)\quad F\left[\begin{matrix}-n : a,b ; c,d ; \\ f : \quad g ; \quad h ;\end{matrix}\; 1,1\right]$$

$$= \frac{(f-a)_n\,(g-a)_n}{(f)_n\,(g)_n}\; {}_4F_3\left[\begin{matrix}-n,c,d,\ f-a-b ; \\ h,\ f-a,\ f-b ;\end{matrix}\; 1\right],$$

provided, $f+g=1+a+b-n$. The above result is due to Srivastava and Saran [110, (2.2)].

Again with a slight adjustment (3.4.5) can also be written as

$$(3.4.7)\quad F\left[\begin{matrix}-n : a, f+g-a+n-1 ; (c) ; \\ f : \qquad g ; (h) ;\end{matrix}\; 1,z\right]$$

$$= \frac{(f-a)_n\,(g-a)_n}{(f)_n\,(g)_n}\; {}_{C+3}F_{H+2}\left[\begin{matrix}-n,a,1-n-g,\ (c) ; \\ f-a,1+a-g-n,(h) ;\end{matrix}\; z\right],$$

for $f+g=1+a+b-n$, which is a generalization of a result established by Srivastava and Saran [110], using a different method.

Equation (3.4.4), after adjusting the parameters becomes

$$(3.4.8)\quad F\left[\begin{matrix}-n : \quad a,h \quad ; f-a,\ f-b,(c) ; \\ f : 1+a+b-f-n ; \ f-a-b,\ (h) ;\end{matrix}\; 1,z\right]$$

$$= \frac{(f-a)_n\,(f-b)_n}{(f)_n\,(f-a-b)_n}\; {}_{C+1}F_H\left[\begin{matrix}-n,(c) ; \\ (h) ;\end{matrix}\; z\right].$$

Further substituting $C=H=3$, $c_1=c$, $c_2=c/2+1$, $c_3=c+h+1$, $h_1=1+c+n$, $h_2=c/2$, $h_3=-h$, $z=-1$ and using the slightly modified summation formula [104,(III.11), p.244]

$$ {}_4F_3\left[\begin{matrix}-n, c, \ c/2+1, \ c+h+1 \ ; \\ 1+c+n, \ c/2, \ -h \ ;\end{matrix} \ -1\right] = \frac{(-)^n \ \Gamma(1+h-n)\,(1+c)_n}{\Gamma(1+h)} , $$

we obtain the following summation formula for $F^{(2)}[1,-1]$

$$ (3.4.9) \quad F\left[\begin{matrix}-n : a, b \ ; \ f-a, f-b, c, c/2+1, 1+c+h \ ; \\ f : 1+a+b-f-n ; \ f-a-b, \ 1+c+n, \ c/2, \ -h \ ;\end{matrix} \ 1,1\right] $$

$$ = \frac{(-)^n \ \Gamma(1+h-n)\,(1+c)_n (f-a)_n (f-b)_n}{\Gamma(1+h) \ (f)_n \ (f-a-b)_n} . $$

(ii) From Saalschütz theorem, we have

$$ {}_3F_2\left[\begin{matrix}-n+u, \ a+u, \ b \ ; \\ f+u, \ 1+a+b-f-n+u \ ;\end{matrix} \ 1\right] = \frac{(f-a)_{n-u} \ (f+u-b)_{n-u}}{(f+u)_{n-u} \ (f-a-b)_{n-u}} . $$

Now applying (3.4.2), we obtain the transformation

$$ (3.4.10) \quad F\left[\begin{matrix}-n, \ a \ : b \ ; \ (c) \ ; \\ f, 1+a+b-f-n \ : \ - \ ; \ (h) \ ;\end{matrix} \ 1, z\right] $$

$$ = \frac{(f-a)_n \ (f-b)_n}{(f)_n \ (f-a-b)_n} \ {}_{C+2}F_{H+2}\left[\begin{matrix}-n \ , \ a, \ (c) \ ; \\ f-b, 1+a-f-n, (h) \ ;\end{matrix} \ z\right] . $$

which is believed to be new. By specializing the parameters in (3.4.10) a number of transformations and summation formulas can be obtained.

3.5 <u>Theorem III</u> : If

$$(3.5.1)\quad {}_{A+B}F_{F+G}\left[\begin{matrix}(a)+u,(b);\\(f)+u,(g);\end{matrix}\;x\right]=\frac{y\Gamma((d)+u)}{\Gamma((j)+u)}\;{}_{E+P}F_{K+Q}\left[\begin{matrix}(e)+u,(p);\\(k)+u,(q);\end{matrix}\;z\right],$$

where y and z are some functions of x but independent of u, then

$$(3.5.2)\quad F\left[\begin{matrix}(a)&:&(b)&;&-m,&(c)&;\\(f)&:&(g)&;&&(h)&;\end{matrix}\;x,v\right]$$

$$=\frac{y\Gamma((d))}{\Gamma((j))}\;F\left[\begin{matrix}(e):(p);&-m,(a),(c),(d),(k)&;\\(k):(q);&(f),(h),(j),(e)&;\end{matrix}\;z,v\right]$$

<u>Proof</u> : From equation (3.5.1), one can easily write

$$\Delta_u^m\left[\frac{\Gamma((a)+u)\Gamma((c)+u)v^u}{\Gamma((f)+u)\Gamma((h)+u)}\;{}_{A+B}F_{F+G}\left[\begin{matrix}(a)+u,&(b)&;\\(f)+u,&(g)&;\end{matrix}\;x\right]\right]$$

$$\Delta_u^m\left[\frac{\Gamma((a)+u)\Gamma((c)+u)\Gamma((d)+u)yv^u}{\Gamma((f)+u)\Gamma((h)+u)\Gamma((j)+u)}\;{}_{E+P}F_{K+Q}\left[\begin{matrix}(e)+u,(p)&;\\(k)+u,(q)&;\end{matrix}\;z\right]\right].$$

The application of Rodrigues type of formula (3.2.1), gives the transformation (3.5.2). Hence theorem III is proved.

Special Cases of Theorem III : (i) Taking $b=b+u$ and $c=c+u$ in the Eulers transformation [85, p.60, (4)], we have

$$ {}_2F_1\left[\begin{matrix} a, b+u ; \\ c+u ; \end{matrix} z\right] = (1-z)^{-a} \; {}_2F_1\left[\begin{matrix} a, c-b ; \\ c+u ; \end{matrix} \frac{-z}{1-z}\right], $$

$$ (3.5.3) \quad F\left[\begin{matrix} b : a ; -m, (d) ; \\ c : - ; \quad (h) ; \end{matrix} z, v\right] $$

$$ = (1-z)^{-a} \; F\left[\begin{matrix} - : a, c-b ; -m, b, (d) ; \\ c : \quad - ; \quad (h) ; \end{matrix} \frac{-z}{1-z}, v\right], $$

(3.5.3) is an extension of the following result (taking $D=0$ and $H=0$ in (3.5.3))

$$ F_1[b, a, -m; c; z, v] = (1-z)^{-a} \; F_3[a, -m, c-b; b; c; \frac{-z}{1-z}, v], $$

established by Srivastava and Singhal [121].

(ii) The Kummers' first formula [85, p.125] on replacing a by $a+u$ and b by $b+u$, gives

$$ {}_1F_1(a+u; b+u; z) = e^z \; {}_1F_1(b-a; b; -z), $$

the above result on using theorem III, yields

$$ (3.5.4) \quad F\left[\begin{matrix} a: - ; -m, (d) ; \\ b: - ; \quad (h) ; \end{matrix} z, v\right] = e^z \; F\left[\begin{matrix} -: b-a ; -m, a, (d) ; \\ b: \quad - ; \quad (h) ; \end{matrix} -z, v\right]. $$

For $D=H=0$, (3.5.4), gives an interesting result

$$(3.5.5) \quad \phi_1[a,-m;b;v,z] = e^z \; \Xi_1[-m,b-a,a;b;v,-z].$$

(iii) Adjusting the parameters in the well known transformation of Agrawal [2,(6A)]

$$(3.5.6) \quad {}_3F_2\left[\begin{matrix}-n,a,b; \\ c,d;\end{matrix}\; 1\right] = \frac{(d-a)_n}{(d)_n}\; {}_3F_2\left[\begin{matrix}-n,a,c-b; \\ c,1+a-d-n;\end{matrix}\; 1\right],$$

and theorem III a number of transformations can be obtained. Since it is not possible to give all the transformations here, a few of them are given below :

(a) In (3.5.6) replacing a,b,c,d by $a+u$, $b+u$, $c+u$ and $d+u$ respectively and make the use of theorem III, we get

$$(3.5.7) \quad F\left[\begin{matrix}a,\ b: -n\ ;\ -m,\ (g)\ ; \\ c,\ d:\ -\ ;\ \ (h)\ ;\end{matrix}\; 1,v\right]$$

$$= \frac{(d-a)_n}{(d)_n}\; F\left[\begin{matrix}a: -n,\ c-b\ ;\ -m,\ b,\ (g)\ ; \\ c:\ 1+a-d-n;\ \ d+n\ ,\ (h)\ ;\end{matrix}\; 1,v\right].$$

Next in the above transformation (3.5.7), letting $G=2$, $g_1=d+n$, $g_2=c-g$, $H=2$, $h_1=b$, $h_2=1+a-h-m$ and $v=1$, and on the R.H.S. applying the same transformation, we easily get

$$(3.5.8) \quad F\left[\begin{matrix}a,\ b: -n\ ;\ -m,\ d+n,\ c-g\ ; \\ c,\ d:\ -\ ;\ \ b,1+a-h-m\ \ ;\end{matrix}\; 1,1\right]$$

$$= \frac{(d-a)_n \ (h)_m}{(d)_n \ (h-a)_m} \ F\left[\begin{matrix} a, g : -n, h+m, c-b ; -m ; \\ c, h : g, 1+a-d-n ; - ; \end{matrix} \ 1,1\right]$$

Further, if we take $G=H=0$, $d=1+a+b-c-m-n$ and $v=1$ in (3.5.7), we obtain the transformation

$$F\left[\begin{matrix} a, b : -n ; -m ; \\ c, 1+a+b-c-m-m : - ; - ; \end{matrix} \ 1,1\right]$$

$$= \frac{(1+b-c-m-n)_n}{(1+a+b-c-m-m)_n} \ F\left[\begin{matrix} a : -n, c-b ; -m, b ; \\ c : c-b+m ; 1+a+b-c-m ; \end{matrix} \ 1,1\right].$$

By making use of the result (3.3.5), on the R.H.S., we get the following summation formula for $F^{(2)}[1,1]$

$$(3.5.9) \quad F\left[\begin{matrix} a, b : -n; -m ; \\ c, 1+a+b-c-m-n : - ; - ; \end{matrix} \ 1,1\right] = \frac{(c-a)_{m+n} \ (c-b)_{m+n}}{(c)_{m+n} \ (c-a-b)_{m+n}} ,$$

In particular for $c=2a$, it reduces to

$$F\left[\begin{matrix} a, b : -n ; -m ; \\ 2a, 1+b-a-m-n : - ; - ; \end{matrix} \ 1,1\right] = \frac{(2a-b)_{m+n} \ (a)_{m+n}}{(2a)_{m+n} \ (a-b)_{m+n}} ,$$

which was established by Sharma [89] and later on derived by Srivastava [113] also.

(b) In case, we replace a by $a+u$ and d by $d+u$ in (3.5.6) and use the theorem III, we obtain

$$(3.5.10)\quad F\begin{bmatrix} a : -n,\ b\ ;\ -m,\ (g)\ ; & \\ & 1,v \\ d :\ \ c\ ;\ \ (h)\ ; & \end{bmatrix}$$

$$= \frac{(d-a)_n}{(d)_n}\ F\begin{bmatrix} a : -n,\ c-b\ ;\ -m,\ (g)\ ; & \\ & 1,v \\ - : c,\ 1+a-d-n\ ;\ d+n,(h)\ ; & \end{bmatrix},$$

in which the substitution $G=H=1$, $g_1=h-g$, $h_1=h$ and $v=1$, gives

$$(3.5.11)\quad F\begin{bmatrix} a : -n,\ c-b\ ;\ -m,\ h-g\ ; & \\ & 1,1 \\ - : c,\ 1+a-d-n\ ;\ h,\ h+n\ ; & \end{bmatrix}$$

$$= \frac{(d)_n\ (d-a)_m}{(d-a)_n\ (d)_m}\ F\begin{bmatrix} a : -n,\ b\ ;\ -m,\ g\ ; & \\ & 1,1 \\ - : c,\ d+m\ ;\ h,\ 1+a-d-m\ ; & \end{bmatrix}.$$

(c) In (3.5.6) first replacing b and c respectively by b+u and c+u, and using theorem III, we get

$$(3.5.12)\quad F\begin{bmatrix} b : -n,\ a\ ;\ -m,\ (g)\ ; & \\ & 1,v \\ c :\ \ d\ ;\ \ (h)\ ; & \end{bmatrix}$$

$$= \frac{(d-a)_n}{(d)_n}\ F\begin{bmatrix} - : -n,\ a,\ c-b\ ;\ -m,\ b,\ (g)\ ; & \\ & 1,v \\ c :\ 1+a-d-n\ ;\ \ (h)\ ; & \end{bmatrix}.$$

Setting $G=H=1$, $g_1=g$, $h_1=1+g+h-m$ and $v=1$ in (3.5.12) and again using (3.5.12) on L.H.S. we obtain the following transformation

$$(3.5.13)\quad F\left[\begin{matrix} - : & -n,\ a,\ c-b\ ; & -m,\ b,\ g\ ; \\ c : & 1+a-d-n\ ; & 1+g-h-m\ ; \end{matrix}\ 1,1\right]$$

$$= \frac{(d)_n\ (h)_m}{(d-a)_n\ (h-g)_m}\ F\left[\begin{matrix} - : & -n,\ a,\ b\ ; & -m,\ g,\ c-b\ ; \\ c : & d\ ; & h\ ; \end{matrix}\ 1,1\right],$$

established earlier by Singal [100, (1.3)].

(iv) By making the use of the transformation due to Bailey [17, 7.2(1)]

$$(3.5.14)\quad {}_4F_3\left[\begin{matrix} -n,\ a,\ b,\ c & ; \\ f,\ g,\ 1+a+b+c-f-g-n & ; \end{matrix}\ 1\right]$$

$$= \frac{(g-c)_n\ (f+g-a-b)_n}{(g)_n\ (f+g-a-b-c)_n}\ {}_4F_3\left[\begin{matrix} -n,\ f-a,\ f-b,\ c & ; \\ f,\ 1-g+c-n,\ f+g-a-b & ; \end{matrix}\ 1\right],$$

in theorem III, a number of transformations for $F^{(2)}[1,v]$ can be derived. As illustration we mention a few of them below.

(a) Replace a and f respectively by $a+u$ and $f+u$ in (3.5.14), then theorem III, gives

$$(3.5.15)\quad F\left[\begin{matrix} a : & -n,\ b,\ c & ; & -m,\ (d)\ ; \\ f : & g,\ 1+a+b+c-f-g-n & ; & (h)\ ; \end{matrix}\ 1,v\right]$$

$$= \frac{(g-c)_n\ (f+g-a-b)_n}{(g)_n\ (f+g-a-b-c)_n}\ F\left[\begin{matrix} f-b : & -n,\ c,\ f-a & ; -m,\ a,\ (d)\ ; \\ f : & 1-g+c-n,\ f+g-a-b & ; f-b,\ (h)\ ; \end{matrix}\ 1,v\right].$$

In particular if we allow $D=H=2$, $d_1=d$, $d_2=f-b$, $h_1=h$, $h_2=1+a+d-b-h-m$ and $v=1$ in (3.5.15), we get the following transformation due to Singal [101, (1.1)] (after using the transformation (3.5.15) on R.H.S.)

$$(3.5.16)\quad F\left[\begin{matrix} a : -n,\ b,\ c \ ;\ -m,\ f-b,\ d\ ; \\ f : g, 1+a+b+c-f-g-n\ ;\ h,\ 1+a+d-b-h-m\ ; \end{matrix}\ 1,1\right]$$

$$= \frac{(g-c)_n\ (f+g-a-b)_n\ (h-d)_m\ (h+b-a)_m}{(g)_n\ (f+g-a-b-c)_n\ (h)_m\ (h+b-a-d)_m}$$

$$X\ \ F\left[\begin{matrix} f-a : -n,\ f-b,\ c\ ;\ -m,\ b,\ d\ ; \\ f\ :\ f+g-a-b, 1+c-g-n;\ h+b-a, 1+h-d-m\ ; \end{matrix}\ 1,1\right].$$

(b) Again replacing c by $c+u$ and g by $g+u$ in (3.5.14) and applying theorem III, we observe that

$$(3.5.17)\quad F\left[\begin{matrix} c : -n,\ a,\ b\ ;\ -m,\ (d)\ ; \\ g : f,\ 1+a+b+c-f-g-n\ ;\ (h)\ ; \end{matrix}\ 1,v\right]$$

$$= \frac{(g-c)_n\ (f+g-a-b)_n}{(g)_n\ (f+g-a-b-c)_n}$$

$$X\ \ F\left[\begin{matrix} c : -n,\ f-a,\ f-b\ ;\ -m,\ f+g-a-b+n,\ (d)\ ; \\ f+g-a-b : f,\ 1-g+c-n\ ;\ g+n,\ (h)\ ; \end{matrix}\ 1,v\right].$$

For $D=H=1$, $d_1=d$ and $h_1=1+c+d-g-m$, the above transformation gives us

$$(3.5.18) \quad F\left[\begin{matrix} c: & -n,\ a,\ b & ; & -m,\ d & ; \\ g: & f,\ 1+a+b+c-f-g-h & ; & 1+c+d-g-m & ; \end{matrix}\ 1,1\right]$$

$$= \frac{(g-c)_n\ (f+g-a-b)_n\ (f+g-a-b-c)_m\ (g-d)_m}{(f+g-a-b-c)_n\ (g)_n\ (g-c-d)_m\ (f+g-a-b)_m}$$

$$\times\ F\left[\begin{matrix} c: & -n, f-a,\ f-b,\ g-d+m\ ; & -m,\ g+n-d,\ a+b-f\ ; \\ g-d: & f, 1-g+c-n, f+g-a-b+m; & g+n, 1+a+b+c-m-f-g; \end{matrix}\ 1,1\right],$$

which is believed to be new.

3.6. Theorem IV : If

$$(3.6.1) \quad {}_{1+A+B}F_{G+H}\left[\begin{matrix} -n+u,\ (a)+u,\ (b)\ ; \\ (g)+u,\ (h)\ ; \end{matrix}\ x\right]$$

$$= \frac{((c)+u)_{n-u}\ ((d))_{n-u}}{((j)+u)_{n-u}\ ((k))_{n-u}}\ y\ {}_{1+E+F}F_{L+M}\left[\begin{matrix} -n+u, (e)+u, (f)\ ; \\ (l)+u, (m)\ ; \end{matrix}\ z\right],$$

where y and z are some functions of x but independent of u, then

$$(3.6.2) \quad F\left[\begin{matrix} -n, (a) : (b)\ ; (p)\ ; \\ (g) : (h)\ ; (q)\ ; \end{matrix}\ x,y\right] = \frac{((c))_n\ ((d))_n}{((j))_n\ ((k))_n}\ y\ \times$$

$$\times\ F\left[\begin{matrix} -n, (e) : (f)\ ; & (a), (l), (p), (j),\ 1-(k)-n\ ; \\ (l) : (m)\ ; & (g), (e), (q), (c),\ 1-(d)-n\ ; \end{matrix}\ z, (-)^{D-K} v\right].$$

Proof : From equation (3.6.1), we have

$$(3.6.3)\quad \Delta_u^n \left[\frac{\Gamma((a)+u)\,\Gamma((p)+u)}{\Gamma((g)+u)\,\Gamma((q)+u)}\, v^u \;{}_{1+A+B}F_{G+H}\left[\begin{matrix} -n+u,\,(a)+u,\,(b)\,; \\ (g)+u,\,(h)\,; \end{matrix}\; x\right]\right]$$

$$= \Delta_u^n \left[\frac{\Gamma((a)+u)\,\Gamma((p)+u)}{\Gamma((g)+u)\,\Gamma((q)+u)} \cdot \frac{((c)+u)_{n-u}\;((d))_{n-u}}{((j)+u)_{n-u}\;((k))_{n-u}}\, y v^u\right.$$

$$\left. \times\; {}_{1+E+F}F_{L+M}\left[\begin{matrix} -n+u,\,(e)+u,\;(f)\;; \\ (l)+u,\;(m)\;; \end{matrix}\; z\right]\right],$$

on applying the formula (1.5.5) and then putting $u=0$ on the right hand side it becomes

$$= y\sum_{r=0}^{n} \frac{(-)^r(-n)_r\,\Gamma((a)+r)\,\Gamma((p)+r)\,((c)+r)_{n-r}\;((d))_{n-r}\,v^r}{r!\,\Gamma((g)+r)\,\Gamma((q)+r)\,((j)+r)_{n-r}\;((k))_{n-r}}$$

$$\times \sum_{s=0}^{n-r} \frac{(-n+r)_s\;((e)+r)_s\;((f))_s}{s!\;((l)+r)_s\;((m))_s}\; z^s$$

$$= \frac{(-)^n\,\Gamma((a))\,\Gamma((p))}{\Gamma((g))\,\Gamma((q))} \cdot \frac{((c))_n\;((d))_n\;y}{((j))_n\;((k))_n}$$

$$\times\; F\left[\begin{matrix} -n,\,(e):\,(f)\,;\;\;(a),\,(l),\,(p),\,(j),\,1-(k)-n\,; \\ (l):\,(m)\,;\;\;(g),\,(e),\,(q),\,(c),\,1-(d)-n\,; \end{matrix}\; z,\,(-)^{D-K}\,v\right].$$

But on taking Lim $u \to 0$ and using the Rodriques' type formula (3.2.2), the L.H.S. of equation (3.6.3) becomes

$$= \frac{(-)^n \Gamma((a))\Gamma((p))}{\Gamma((g))\Gamma((q))} F\left[\begin{matrix} -n,(a) & : & (b) & ; & (p) & ; \\ (g) & : & (h) & ; & (q) & ; \end{matrix} x,v\right].$$

Hence, we get the required transformation (3.6.2).

Special Cases of Theorem IV : From (3.5.6) and (3.5.14), theorem IV gives a number of transformations, a few of them are given below without proof (as the proof is similar to the above we used in case of theorem III).

$$(3.6.4) \quad F\left[\begin{matrix} -n,a:b;(g); \\ d:c;(h); \end{matrix} 1,v\right] = \frac{(d-a)_n}{(d)_n} F\left[\begin{matrix} -n,a:c-b;(g); \\ 1+a-d-n:c;(h); \end{matrix} 1,-v\right],$$

$$(3.6.5) \quad F\left[\begin{matrix} -n, & b & : & a & ; & (g) & ; \\ & c & : & d & ; & (h) & ; \end{matrix} 1,v\right]$$

$$= \frac{(d-a)_n}{(d)_n} F\left[\begin{matrix} -n & : & a, c-b & ; & b, 1-d-n,(g) & ; \\ c,1+a-d-n & : & - & ; & (h) & ; \end{matrix} 1,v\right],$$

$$(3.6.6) \quad F\left[\begin{matrix} -n & : & a, & b & ; & (g) & ; \\ c & : & & d & ; & (h) & ; \end{matrix} 1,v\right]$$

$$= \frac{(d-a)_n}{(d)_n} F\left[\begin{matrix} -n, c-b & : & a & ; & 1-d-n,(g) & ; \\ c, 1+a-d-n & : & - & ; & c-b,(h) & ; \end{matrix} 1,v\right],$$

$$(3.6.7) \quad F\left[\begin{matrix} -n & : & a, & b & ; & (g) & ; \\ d & : & & c & ; & (h) & ; \end{matrix} 1,v\right]$$

$$= \frac{(d-a)_n}{(d)_n} F\left[\begin{array}{l} -n : a,\ c-b \quad ; \quad (g) \ ; \\ - : c, 1+a-d-n \ ; \ d-a, (h) \ ; \end{array} 1, v\right],$$

$$(3.6.8) \quad F\left[\begin{array}{l} -n,\ a : b \ ; \ g \ ; \\ \quad d : c \ ; \ h \ ; \end{array} 1, -1\right] = F\left[\begin{array}{l} -n, a : c-b \ ; \ h-g \ ; \\ \quad d : \quad c \ ; \quad h \ ; \end{array} -1, 1\right],$$

$$(3.6.9) \quad F\left[\begin{array}{l} \qquad -n : a,\ c-b \ ; \ b,\ 1-d-n,\ g \ ; \\ c,\ 1+a-d-n : \quad - \ ; \qquad h \ ; \end{array} 1, 1\right]$$

$$= \frac{(d)_n \ (h-g)_n}{(d-a)_n \ (h)_n} F\left[\begin{array}{l} -n \qquad : b,\ 1-h-n,\ a \ ; \ g, c-b \ ; \\ c, 1+g-h-n : \qquad d \ ; \quad - \ ; \end{array} 1, 1\right],$$

$$(3.6.10) \quad F\left[\begin{array}{l} \quad -n,\ c-b : a \ ; \ 1-d-n,\ f,\ g \ ; \\ c,\ 1+a-d-n : \qquad c-b, \quad h \ ; \end{array} 1, 1\right]$$

$$= \frac{(d)_n \ (h-f)_n}{(d-a)_n \ (h)_n} F\left[\begin{array}{l} -n,\ c-g \quad : 1-h-n,\ a,\ b \ ; \ f \ ; \\ c,\ 1+f-h-n : \quad c-g, \quad d \ ; \ - \ ; \end{array} 1, 1\right],$$

$$(3.6.11) \quad F\left[\begin{array}{l} -n : \quad a,\ c-b \quad ; \quad f,\ g \ ; \\ - : c,\ 1+a-d-n \ ; \ d-a,\ h \ ; \end{array} 1, 1\right]$$

$$= \frac{(d-f)_n}{(d-a)_n} F\left[\begin{array}{l} -n : a,\ b \quad ; \ f, \quad h-g \ ; \\ - : d-f,\ c \ ; \ h,\ 1+f-d-n \ ; \end{array} 1; 1\right],$$

$$(3.6.12) \quad F\left[\begin{array}{l} -n : \quad a,\ b,\ c \qquad ; (d) \ ; \\ f : g, 1+a+b+c-f-g-h; (h) \ ; \end{array} 1, v\right] = \frac{(g-c)_n \ (f+g-a-b)_n}{(g)_n \ (f+g-a-b-c)_n}$$

$$\times \; F\left[\begin{matrix} -n,\ f-a,\ f-b & : & c\ ; & 1-g-n,\ f+g-a-b-c,\ (d)\ ; \\ f,\ 1-g+c-n,\ f+g-a-b & : & -\ ; & f-a,\ f-b,\ (h)\ ; \end{matrix} \; 1,v\right],$$

$$(3.6.13) \quad F\left[\begin{matrix} -n & : & a,\ b,\ c\ ; & (d)\ ; \\ g & : & f,1+a+b+c-f-g-n\ ; & (h)\ ; \end{matrix} \; 1,v\right] = \frac{(g-c)_n \; (f+g-a-b)_n}{(g)_n \; (f+g-a-b-c)_n}$$

$$\times \; F\left[\begin{matrix} -n & : & f-a,\ f-b,\ c\ ; & f+g-a-b-c,\ (d)\ ; \\ f+g-a-b & : & f,\ 1-g+c-n\ ; & g-c,\ (h)\ ; \end{matrix} \; 1,v\right],$$

$$(3.6.14) \quad F\left[\begin{matrix} -n\ ,\ c & : & a,\ b\ ; & (d)\ ; \\ g,\ 1+a+b+c-f-g-n & : & f\ ; & (h)\ ; \end{matrix} \; 1,v\right]$$

$$= \frac{(g-c)_n \; (f+g-a-b)_n}{(g)_n \; (f+g-a-b-c)_n} \; F\left[\begin{matrix} -n,\ c & : & f-a,\ f-b\ ; & (d)\ ; \\ 1-g+c-n,\ f+g-a-b & : & f\ ; & (h)\ ; \end{matrix} \; 1,v\right],$$

$$(3.6.15) \quad F\left[\begin{matrix} -n,\ f-a,\ f-b & : & c\ ; & 1-g-n,\ f+g-a-b-c,\ a',b',c'\ ; \\ f,1-g+c-n,f+g-a-b & : & -\ ; & f-a,f-b,g',1+a'+b'+c'-f-g'-n\ ; \end{matrix} \; 1,1\right]$$

$$= \frac{(g)_n \; (f+g-a-b-c)_n \; (g'-c')_n \; (f+g'-a'-b')_n}{(g-c)_n \; (f+g-a-b)_n \; (g')_n \; (f+g'-a'-b'-c')_n}$$

$$\times \; F\left[\begin{matrix} -n,\ f-a',\ f-b' & : & 1-g'-n,\ f+g'-a'-b'-c',\ a,b,c\ ; & c'\ ; \\ f,\ 1-g'+c'-n,f+g'-a'-b' & : & f-a',f-b',g,\ 1+a+b+c-f-g-n\ ; & -\ ; \end{matrix} \; 1,1\right],$$

$$(3.6.16)\quad F\left[\begin{matrix} -n : & a, b, c ; & d, e ; \\ g : & f, 1+a+b+c-f-g-n ; & 1+d+e-g-n ; \end{matrix} \; 1,1\right]$$

$$= \frac{(g-c)_n \, (f+g-a-b-e)_n \, (g-d)_n}{(g)_n \, (f+g-a-b-c)_n \, (g-d-e)_n}$$

$$\times \; F\left[\begin{matrix} -n : & g-d-e, f-a, f-b, c & ; a+b-f, g-c-d, e & ; \\ g-d : & f+g-a-b-e. f, 1-g+c-n; & g-c, 1+a+b+e-f-g-n & ; \end{matrix} \; 1,1\right]$$

and

$$(3.6.17)\quad F\left[\begin{matrix} -n, b & : a ; & f ; \\ c, 1+f+a+b-c-n & : - ; & - ; \end{matrix} \; 1,1\right] = \frac{(c-b)_n \, (c-a-f)_n}{(c)_n \, (c-a-b-f)_n} .$$

For $c=2b$ equation (3.6.17) reduces to known result of Sharma [89]. Later on the same result was also proved by Srivastava [113] by a different technique.

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CHAPTER - IV

STUDY OF GENERALIZED BASIC HYPERGEOMETRIC SERIES WITH THE HELP OF RODRIGUES' TYPE FORMULAE

1.1 Introduction : Toscano in 1949 [131, (14)] introduced the following difference representation for ordinary generalized hypergeometric series of one variable

$$(4.1.1) \quad {}_{p+1}F_{q+1}\left[\begin{matrix} -n, a_1+u, \cdots, a_p+u \,; \\ u, b_1+u, \cdots, b_q+u \,; \end{matrix} \; t\right]$$

$$= (-)^n \frac{\Gamma(u)\Gamma(b_1+u)\ldots\Gamma(b_q+u)}{\Gamma(a_1+u)\ldots\Gamma(a_p+u)} t^{-u} \Delta_u^n \left[\frac{\Gamma(a_1+u)\ldots\Gamma(a_p+u)t^u}{\Gamma(u)\Gamma(b_1+u)\ldots\Gamma(b_q+u)}\right]$$

and derived certain properties of such polynomials. Later, Gasper [45] also obtained certain results for these series. In 1973 Agrawal [2] derived some transformations and in 1974 Agrawal and Manglik [4] obtained three term relations for ordinary hypergeometric series using the same operational technique.

In the present Chapter following Toscano [131], we have introduced some Rodriques' type representations for generalized basic hypergeometric series in term of difference operators $(q^u\Delta_u)$ and Δ'. From these Rodriques' type

formulae we shall obtain certain summation formulas, transformations, generating relations and three term relations for generalized basic hypergeometric series of one variable.

4.2 <u>Preliminary Definitions and Results</u> : In deducing our results we shall make the use of the following notations and definitions :

(4.2.1) $(a;q)_n = (1-q^a)(1-q^{a+1})\dots(1-q^{a+n-1})/(1-q)^n;\ n=1,2,\dots,$

$(a;q)_0 = 1\ ; \qquad |q| < 1\ ,$

$$(a;q)_{n-r} = \frac{(a;q)_n q^{r(r+1)/2}}{(q^{1-n}\cancel{/a};q)_r.\ (-a)^r\ q^{nr}}\ ,$$

q^{1-n-a}

$\underline{q^n}$

$$(aq^r;q)_{n-r} = \frac{(a;q)_n}{(a;q)_r}\ ,$$

r/

for all values of a, real or complex.

(4.2.2) $\Gamma_q(qv) = (v;q)\Gamma_q(v) = \underline{[(1-q^a)/(1-q)]\Gamma_q(v)}$, what's a ?

$\Gamma_q(vq^n)/\Gamma_q(v) = (v;q)_n$.

Also for our convenience, we shall write $\Gamma_q((a)u,(b)u,\dots)$ in place of $\Gamma_q((a)u)\Gamma_q((b)u)\dots,$

where $\Gamma_q((a)u) \equiv \Gamma_q(a_1u,\dots,a_Au)$.

$$(4.2.3)\quad (q*x+y)^n = (xq^{(n-1)/2}+y)\ (xq^{(n-2)/2}+yq^{1/2})\ldots$$

$$\times\ (x+yq^{(n-1)/2});\quad n = 1,2,3,\cdots,$$

which for $y = 0$, gives

$$(4.2.4)\quad (q*x)^n = x^n\ q^{n(n-1)/4}$$

By mathematical induction, we also have

$$(4.2.5)\quad (q*x+y)^n = \sum_{r=0}^{n} \left[\begin{matrix} n; \\ r; \end{matrix}\ q\right] (q*x)^{n-r}\ (q*y)^r\ ,$$

where

$$(4.2.6)\quad \left[\begin{matrix} n; \\ r; \end{matrix}\ q\right] \equiv \left[\begin{matrix} n \\ r \end{matrix}\right] = \frac{(q;q)_n}{(q;q)_r\ (q;q)_{n-r}}$$

$$= (-)^r\ q^{-r(r-1-2n)/2}\ \frac{(q^{-n};q)_r}{(q;q)_r}\ .$$

It is known as the q-binomial coefficient.

The basic exponential functions are defined by means of

$$(4.2.7)\quad e_q(x) = \sum_{r=0}^{\infty} \frac{x^r}{(q,q)_r}$$

$$(4.2.8)\quad E(q,x) = \sum_{r=0}^{\infty} \frac{x^r q^{r(r-1)/4}}{(q;q)_r} = \sum_{r=0}^{\infty} \frac{(q*x)^r}{(q;q)_r}\ .$$

We also make the use of the operators $(q^u\Delta_u)$ and Δ' which have following operational relations (for positive integer n)

$$(4.2.9)\quad \Delta_u f(u) = f(u+1) - f(u)$$

$$(4.2.10)\quad (q^u\Delta_u)^n f(u)=q^{un} \sum_{r=0}^{n} (-)^{n-r}\left[{n \atop r}\right]q^{r(r-1)/2} f(u+r) ,$$

$$(4.2.11)\quad (q^u\Delta_u)^n[f(u)g(u)] = \sum_{r=0}^{n} \left[{n \atop r}\right] (q^n\Delta_u)^{n-r} f(u+r)$$

$$\times\ (q^u\Delta_u)^r g(u) ,$$

$$(4.2.12)\quad (\Delta')^{n+1} f(u) = q^n(\Delta')^n f(u) - (\Delta')^n f(u+1) ,$$

$$(4.2.13)\quad (\Delta')^n f(u) = \sum_{r=0}^{n} \frac{(q^{-n};q)_r}{(q;q)_r} q^{n(n-1)/2+r} f(u+r)$$

and

$$(4.2.14)\quad (\Delta')^n [f(u)\ g(u)] = \sum_{r=0}^{n} \left[{n \atop r}\right](\Delta')^{n-r} f(u+r) (\Delta')^r g(u) ,$$

(these can be easily verified by mathematical induction).

We shall consider the following generalized basic hypergeometric series of one variable, defined as

(4.2.15) $$ {}_A\phi_B((a);(b);q,z) \equiv {}_A\phi_B\left[\begin{matrix}(a) & ; \\ & q,z \\ (b) & ;\end{matrix}\right] = \sum_{n=0}^{\infty} \frac{((a);q)_n}{((b);q)_n} \frac{z^n}{(q,q)_n} $$

and

$$ {}_A\phi_B^{*}((a);(b);q,z) = \sum_{n=0}^{\infty} \frac{((a);q)_n}{((b);q)_n (q;q)_n} z^n q^{n(n-1)/2} , $$

where (a) abbreviates the sequence of A parameters $a_1,\ldots,a_A$ and $((a);q)_n \equiv (a_1;q)_n \ldots (a_A;q)_n$; with similar interpretations for (b).

Now, we give the following Rodrigues' type formulae for generalized basic hypergeometric series and other relations, which shall be useful in subsequent sections of this chapter :

(4.2.16) $$ {}_{A+1}\phi_B(q^{-n},(a)u;(b)u;q,x) $$

notation?

$$ = (-)^n \frac{\Gamma_q((b)u)}{\Gamma_q((a)u)} x^{-u} (q^u\Delta_u)^n \left[\frac{\Gamma_q((a)u)}{\Gamma_q((b)u)} q^{-nu} x^u \right] , $$

(4.2.17) $$ \lim_{u\to 0} \left[(q^u\Delta_u)^n \left\{ \frac{\Gamma_q((a)u)}{\Gamma_q((b)u)} q^{-nu} x^u \right\} \right] $$

$$ = (-)^n \frac{\Gamma_q((a))}{\Gamma_q((b))} {}_{A+1}\phi_B(q^{-n},(a);(b);q,x) , $$

(4.2.18) $$ {}_{A+1}\phi_B(q^{-n},(a)u;(b)u;\ q,\ qx) $$

$$= \frac{\Gamma_q((b)u)}{\Gamma_q((a)u)} q^{-n(n-1)/2} x^{-u} (\Delta')^n \left[\frac{\Gamma_q((a)u)}{\Gamma_q((b)u)} \right]$$

(4.2.19) $$(q^u \Delta_u)^n \left[\frac{\Gamma_q(au)}{\Gamma_q(eu)} q^{(e-a)u} \right] = (-)^n q^{(n+e-a)u} \frac{\Gamma_q((a)u)(e/a;q)_n}{\Gamma_q(eu)(eu;q)_n},$$

(4.2.20) $$(\Delta')^n \left[\frac{\Gamma_q(au)}{\Gamma_q(eu)} \right] = \frac{\Gamma_q(au)(e/a;q)_n}{\Gamma_q(eu)(eu;q)_n} q^{(a+u)n+n(n-1)/2},$$ e−a

eth

(4.2.21) $$E(q,x+y) = E(q,x)\, E(q,y)$$

and

(4.2.22) $$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} (q^u \Delta_n)^n f(u)$$

$$= e_q(-zq^u) \sum_{r=0}^{\infty} \frac{z^r q^{ru+r(r-1)/2}}{(q;q)_r} f(u+r) .$$

Proof of (4.2.16), (4.2.17) and (4.2.18). In view of the operational relation (4.2.10), we have

$$(q^u \Delta_u)^n \left[\frac{\Gamma_q((a)u)}{\Gamma_q((b)u)} q^{-nu} x^u \right]$$

$$= \sum_{r=0}^{n} (-)^{n-r} \begin{bmatrix} n \\ r \end{bmatrix} \frac{\Gamma_q((a)uq^r)}{\Gamma_q((b)uq^r)} q^{r(r-1)/2-nr} x^{u+r}$$

$$= (-)^n x^u \frac{\Gamma_q((a)u)}{\Gamma_q((b)u)} \sum_{r=0}^{n} \frac{(q^{-n};q)_r \; ((a)u;q)_r}{(q;q)_r \; ((b)u;q)_r} x^r ,$$

which by the definition (4.2.15), gives us the required Rodrigues' type formulas (4.2.16) and (4.2.17).

In a similar manner, by applying (4.2.13) on the R.H.S. of (4.2.18), we can easily get the another Rodrigues' type formula (4.2.18).

<u>Proof of (4.2.19) and (4.2.20)</u>. From (4.2.16), we have

$$(q^u\Delta_u)^n \left[\frac{\Gamma_q(au)}{\Gamma_q(eu)} q^{(e-a)u} \right] \qquad q^{-nu}$$

$$= (-)^n q^{(n+e-a)u} \frac{\Gamma_q(au)}{\Gamma_q(eu)} {}_2\phi_1(q^{-n}, au; \cancel{bu}; q, q^{n+e-a}) , \qquad e$$

which with the help of well known summation formula [44, (1.4.3.12), p.28]

$${}_2\phi_1(q^{-n}, b; d; q, q^{n+d-b}) = (d/b;q)_n/(d;q)_n , \qquad \text{wrong notation } (d-b;q)_n$$

yields (4.2.19).

While the use of (4.2.13) and the q-analogue of Vandermonde's theorem [44, (1.4.3.11), p.28]

$${}_2\phi_1(q^{-n}, b; d; q, q) = a^{bn} (d/b;q)_n/d;q)_n ,$$

gives required result (4.2.20).

It is necessary to point out here that the results (4.2.19) and (4.2.20) can also be proved by mathematical induction methods.

Proof of (4.2.21) and (4.2.22). In view of the definitions (4.2.8) and (4.2.5), it follows that

$$E(q,x+y) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \left[{n \atop r} \right] \frac{(q*x)^{n-r} (q*y)^{r}}{(q;q)_n}$$

$$= \sum_{n,r=0}^{\infty} \frac{(q*x)^n}{(q;q)_n} \frac{(q*y)^r}{(q;q)_r}$$

$$= E(q,x)\ E(q,y)\ ,$$

which completes the proof of (4.2.21).

To prove (4.2.22) consider

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} (q^u \Delta_u)^n f(u)$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} q^{nu} \sum_{r=0}^{n} (-)^{n-r} \left[{n \atop r} \right] q^{r(r-1)/2} f(u+r)$$

$$= \sum_{n,r=0}^{\infty} \frac{(-)^n z^{n+r} q^{(n+r)u}}{(q;q)_n (q;q)_r} q^{r(r-1)/2} f(u+r)$$

$$= e_q(-zq^u) \sum_{r=0}^{\infty} \frac{z^r}{(q;q)_r} q^{ru+r(r-1)/2} f(u+r).$$

Hence result (4.2.22) is proved.

4.3 <u>A Simple Proof of Bailey's Theorem.</u> The Bailey's theorem [16,(c), p.512] is

$$(4.3.1) \qquad {}_4F_3\left[\begin{matrix} -n,\ b+n,\ a/2,\ a/2+1/2\ ; \\ a+1,\ b/2,\ b/2+1/2\ ; \end{matrix}\ 1\right] = \frac{(b-a)_n}{(b)_n}\ .$$

We give a simple proof of the above theorem, which is believed to be new, as follows.

Consider

$$\Delta_u\left[\frac{\Gamma(b+1+u,\ a/2+u,\ a/2+1/2+u)}{\Gamma(a+1+u,\ b/2+u,\ b/2+1/2+u)}\right]$$

$$= \frac{\Gamma(b+2+u, a/2+1+u, a/2+3/2+u)}{\Gamma(a+2+u, b/2+1+u, a/2+3/2+u)} - \frac{\Gamma(b+1+u, a/2+u, a/2+1/2+u)}{\Gamma(a+1+u, b/2+u, b/2+1/2+u)}$$

or

$$\frac{\Gamma(a+1+u, b/2+u, b/2+1/2+u)}{\Gamma(\cancel{a}+1+u, \cancel{b}/2+u, \cancel{b}/2+1/2+u)}\ \Delta_u\left[\frac{\Gamma(b+1+u, a/2+u, a/2+1/2+u)}{\Gamma(a+1+u, b/2+u, b/2+1/2+u)}\right]$$

b

a

a

$$= \frac{\Gamma(b+1+u)(a/2+u)(a/2+1/2+u)}{\Gamma(a+1+u)(b/2+u)(b/2+1/2+u)} - 1\ .$$

9 9

From (4.1.1) with $u=0$, we get

$$(4.3.2) \qquad {}_4F_3\left[\begin{matrix} -1,\ b+1,\ a/2,\ a/2+1/2\ ; \\ a+1,\ b/2,\ b/2+1/2\ ; \end{matrix}\ 1\right] = \frac{(b-a)_1}{(b)_1}\ .$$

In a similar manner, we observe that

$$ {}_4F_3\left[\begin{matrix} -r,\ b+r,\ a/2,\ a/2+1/2\ ; \\ a+1,\ b/2,\ b/2+1/2\ ; \end{matrix}\ 1\right] = \frac{(b-a)_r}{(b)_r}, $$

for $r=2,3,\ldots,$. Hence by mathematical induction theorem (4.3.1) is proved.

By employing the above technique a number of well known summation theorems for ordinary hypergeometric series can be proved very easily.

4.4 An Alternate Proof of A Summation Formula For ${}_6\Phi_5$.

We have the following summation formula [104, (3.3.1.4), p.96] for a well-poised series ${}_6\Phi_5$,

$$ (4.4.1)\quad {}_6\Phi_5\left[\begin{matrix} q^{-n}, a, q\sqrt{a}, -q\sqrt{a}, b, c\ ; \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{1+n}\ ; \end{matrix}\ q, q^{1+n+a-b-c}\right] $$

$$ = \frac{(aq;q)_n\ (aq/bc;q)_n}{(aq/b;q)_n\ (aq/c;q)_n}, $$

We shall give an alternative proof of this summation formula.

From (4.2.9), we have

$$ (q^u\Delta_u)\left[\frac{\Gamma_q(au,\ qu\sqrt{a},\ -qu\sqrt{a},\ bu,\ cu)\ q^{(1+a-b-c)u}}{\Gamma_q(u\sqrt{a},\ -u\sqrt{a},\ auq/b,\ auq/c,\ auq^2)}\right] $$

$$= \frac{\Gamma_q(au,\ qu\sqrt{a},\ -qu\sqrt{a},\ bu,\ cu)\ q^{(2+a-b-c)u}}{\Gamma_q(u\sqrt{a},\ -u\sqrt{a},\ auq/b,\ auq/c,\ auq^2)}$$

$$\times \Big[\frac{(1-q^{a+u})(1-q^{1+a/2+u})(1+q^{1+a/2+u})(1-q^{b+u})(1-q^{c+u})}{(1-q^{a/2+u})(1+q^{a/2+u})(1-q^{1+a-b+u})(1-q^{1+a-c+u})(1-q^{2+a+u})}$$

$$\times\ q^{1+a-b-c} - 1 \Big] .$$

Therefore

$$(-)\ \frac{\Gamma_q(\sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ aq^2)}{\Gamma_q(a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c)}$$

$$\times \lim_{u\to 0}\Big[(q^u\Delta_u)\ \frac{\Gamma_q(au, qu\sqrt{a}, -qu\sqrt{a},\ bu,\ cu)\ q^{(1+a-b-c)u}}{\Gamma_q(u\sqrt{a}, -u\sqrt{a}, aqu/b,\ aqu/c,\ auq^2)}\Big]$$

$$= \frac{(1-q^{1+a})\ (1-q^{1+a-b-c})}{(1-q^{1+a-b})\ (1-q^{1+a-c})} = \frac{(aq;q)\ \ (aq/bc;q)}{(aq/b;q)\ \ (aq/c;q)}\ ,$$

In a similar manner we see that

$$\frac{(-)^n\ \Gamma_q(\sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ aq^3)}{\Gamma_q(a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c\ )}$$

$$= \lim_{u\to 0}\Big[(q^u\Delta_u)^2\Big[\frac{\Gamma_q(au, qu\sqrt{a}, -qu\sqrt{a},\ bu,\ cu)q^{(1+a-b-c)u}}{\Gamma_q(u\sqrt{a},\ -u\sqrt{a},\ aqu/b,\ aqu/c,\ auq^3\ )}\Big]$$

$$= \frac{(aq;q)_2\ \ (aq/bc)_2}{(aq/b)_2\ \ (aq/c)_2}\ .$$

Hence in general we can write that if n is any positive integer (by mathematical induction)

$$(4.4.2) \quad \frac{(-)^n \Gamma_q(\sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{1+n})}{\Gamma_q(a, q\sqrt{a}, -q\sqrt{a}, b, c)}$$

$$\times \lim_{u\to 0}\left[(q^u \Delta_u)^n \frac{\Gamma_q(au, qu\sqrt{a}, -qu\sqrt{a}, bu, cu)\, q^{(1+a-b-c)u}}{\Gamma_q(u\sqrt{a}, -u\sqrt{a}, auq/b, auq/c, auq^{1+n})}\right]$$

$$= \frac{(aq;q)_n \ (aq/bc;q)_n}{(aq/b;q)_n \ (aq/c;q)_n} .$$

Applying the Rodrigues' type formula (4.2.17) on the L.H.S. of (4.4.2), we get the summation formula (4.4.1).

4.5 Certain Transformations : Through operational technique following transformations have been derived for basic hypergeometric series of one variable. Some of these results were obtained by Askey and Wilson [15a] by using a different technique and others thus obtained provide, the q-analogues of the known results which otherwise are not easily derivable :

$$(4.5.1) \quad {}_3\phi_2\left[\begin{matrix} q^{-n}, a, b\ ; \\ e, f\ ; \end{matrix}\ q, q^{n+e+f-a-b}\right]$$

$$= \frac{(e/a;q)_n}{(e;q)_n}\ {}_3\phi_2\left[\begin{matrix} q^{-n}, a, f/b\ ; \\ f, aq^{1-n}/e\ ; \end{matrix}\ q, q\right] ;$$

(q-analogue of a result due to Agrawal [2, (6A)]) ,

(4.5.2) $${}_3\phi_2\left[\begin{matrix} q^{-n}, a, b ; \\ e, f ; \end{matrix} q, q\right]$$

$$= q^{an} \frac{(f/a;q)_n}{(f;q)_n} \; {}_3\phi_2\left[\begin{matrix} q^{-n}, a, e/b ; \\ e, aq^{1-n}/f ; \end{matrix} q; q^{1+b-f}\right]$$

(due to Askey and Wilson [15a, (1.30)]) ,

(4.5.3) $${}_3\phi_2\left[\begin{matrix} q^{-n}, a, b ; \\ e, f ; \end{matrix} q, q^{e+f+n-a-b}\right]$$

$$= q^{an} \frac{(e/a;q)_n \; (f/a;q)_n}{(e;q)_n \; (f;q)_n} \; {}_3\phi_2\left[\begin{matrix} q^{-n}, a, abq^{1-n}/ef ; \\ aq^{1-n}/e, aq^{1-n}/f ; \end{matrix} q, q^{1-b}\right] ;$$

(q-analogue of another result due to Agrawal [2, (6B)]),

(4.5.4) $${}_3\phi_2\left[\begin{matrix} q^{-n}, a, b ; \\ e, f ; \end{matrix} q, q^{e+f+n-a-b}\right]$$

$$= \frac{(ef/ab;q)_n}{(e;q)_n} \; {}_3\phi_2\left[\begin{matrix} q^{-n}, f/a, f/b ; \\ f, ef/ab ; \end{matrix} q, q^{n+e}\right] ,$$

(4.5.5) $${}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c ; \\ e, f, g ; \end{matrix} q, q\right]$$

$$= q^{an} \frac{(e/a;q)_n \; (f/a;q)_n}{(e;q)_n \; (f;q)_n} \; {}_4\phi_3\left[\begin{matrix} q^{-n}, a, g/b, g/c ; \\ g, aq^{1-n}/e, aq^{1-n}/f ; \end{matrix} q, q\right] ;$$

provided $e+f+g = 1+a+b+c-n$,

(4.5.6) $${}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & b, & c & ; \\ f, & g, & abcq^{1-n}/fg & & ; \end{matrix} \; q,q\right]$$

$$= \frac{(f/a;q)_n \; (fg/bc;q)_n}{(f;q)_n \; (fg/abc;q)_n} \; {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & g/b, & g/c & ; \\ g, aq^{1-n}/f, & fg/bc & & & ; \end{matrix} \; q,q\right],$$

(basic analogue of a result of Bailey [17,7.2(1)])

(4.5.7) $${}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & b, & c & ; \\ & e, & f, & g & ; \end{matrix} \; q,q\right] = q^{(b+c-g)n} \frac{(aq^{1-n}/e;q)_n}{(e;q)_n}$$

$$\times \frac{(aq^{1-n}/f;q)_n}{(f;q)_n} \; {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & g/b, & g/c & ; \\ g, & aq^{1-n}/e, & aq^{1-n}/f & & ; \end{matrix} \; q,q\right];$$

when $1-n+a+b+c = e+f+g$,

(due to Askey and Wilson [15a,(1.28)])

and

(4.5.8) $$e_q(-z) \; {}_1\phi_1^{*}(a;b;q,zq^{b-a}) = {}_1\phi_1(b/a;b;q,-z)$$

(basic-analogue of Kummer's first formula [85])

Proof of (4.5.1), (4.5.2), (4.5.3) and (4.5.4) :

From (4.2.17), we observe that

(4.5.9) $$\lim_{u\to 0}\left[(q^u\Delta_u)^n \frac{\Gamma_q(au,bu)}{\Gamma_q(eu,fu)} q^{(e+f-a-b)u}\right]$$

$$= (-)^n \frac{\Gamma_q(a,b)}{\Gamma_q(e,f)} \ {}_3\phi_2 \left[\begin{matrix} q^{-n}, & a, & b \ ; \\ & e, & f \ ; \end{matrix} \ q, q^{n+e+f-a-b} \right].$$

Alternatively the application of (4.2.11), gives

$$(q^u \Delta_u)^n \left[\left(\frac{\Gamma_q(au)}{\Gamma_q(eu)} q^{(e-a)u} \right) \left(\frac{\Gamma_q(bu)}{\Gamma_q(fu)} q^{(f-b)u} \right) \right]$$

$$= \sum_{r=0}^{n} (-)^r q^{-r(r-1-2n)/2} \frac{(q^{-n};q)_r}{(q;q)_r}$$

$$\times \ (q^u \Delta_u)^{n-r} \left[\frac{\Gamma_q(auq^r)}{\Gamma_q(euq^r)} q^{(e-a)(u+r)} \right] (q^u \Delta_u)^r \left[\frac{\Gamma_q(bu)}{\Gamma_q(fu)} q^{(f-b)u} \right].$$

on the R.H.S. of the above expression applying (4.2.19), we get

(4.5.10) $$\lim_{u \to 0} \left[(q^u \Delta_u)^n \frac{\Gamma_q(au,bu)}{\Gamma_q(eu,fu)} q^{(e+f-a-b)} \right]$$

$$= \frac{\Gamma_q(b)}{\Gamma_q(f)} \sum_{r=0}^{n} (-)^{n-r} q^{-r(r-1+2n)/2+(e-a)r}$$

$$\times \ \frac{(q^{-n};q)_r \ \Gamma_q(aq^r) \ (e/a;q)_{n-r} \ (f/b;q)_r}{(q;q)_r \ \Gamma_q(eq^r) \ (eq^r;q)_{n-r} \ (f;q)_r}$$

$$= (-)^n \frac{\Gamma_q(a,b) \ (e/a;q)_n}{\Gamma_q(e,f) \ (e;q)_n} \ {}_3\phi_2 \left[\begin{matrix} q^{-n}, \ a, \ f/b \ ; \\ f, \ aq^{1-n}/e \ ; \end{matrix} \ q, q \right].$$

Equating (4.5.9) and (4.5.10), we get the required transformation (4.5.1).

To prove (4.5.2) taking $f(u) = \Gamma_q(au)/\Gamma_q(eu)$ and $g(u) = \Gamma_q(eu)/\Gamma_q(fu)$, in the operational formula (4.2.14) and applying (4.2.20), we get

$$(\Delta')^n \left[\frac{\Gamma_q(au,bu)}{\Gamma_q(eu,fu)} \right] = \sum_{r=0}^{n} \left[{n \atop r} \right] q^{(n-r)(n-r-1)/2+(a+u+r)(n-r)}$$

$$\times \frac{(f/a;q)_{n-r}}{(fuq^r;q)_{n-r}} \, q^{r(r-1)/2+(b+u)r} \, \frac{\Gamma_q(bu)\ (e/b;q)_r}{\Gamma_q(eu)\ (eu;q)_r} ,$$

or

$$(4.5.11) \quad q^{-n(n-1)/2} \frac{\Gamma_q(eu,fu)}{\Gamma_q(au,bu)} (\Delta')^n \left[\frac{\Gamma_q(au,bu)}{\Gamma_q(eu,fu)} \right]$$

$$= q^{(a+u)n} \frac{(f/a;q)_n}{(fu;q)_n} \, {}_3\phi_2 \left[\begin{matrix} q^{-n}, & au, & e/b\ ; \\ eu, & aq^{1-n}/f\ ; & \end{matrix} \ q, q^{1+b-f} \right] .$$

But from (4.2.18), we observe that

$$(4.5.12) \quad q^{-n(n-1)/2} \frac{\Gamma_q(eu,fu)}{\Gamma_q(au,bu)} (\Delta')^n \left[\frac{\Gamma_q(au,bu)}{\Gamma_q(eu,fu)} \right]$$

$$= {}_3\phi_2 \left[\begin{matrix} q^{-n}, & au, & bu\ ; \\ & eu, & fu\ ; \end{matrix} \ q, q \right] .$$

Hence on comparing (4.5.11) and (4.5.12), we get (4.5.2).

In (4.5.2) replacing e and b by 1+a-e-n and f-b respectively, we have

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, \ a, \ f/b \ ; \\ f, \ aq^{1-n}/e \ ; \end{matrix} \ q, q \right] $$

$$ = q^{an} \frac{(f/a;q)_n}{(f;q)_n} \ {}_3\phi_2\left[\begin{matrix} q^{-n}, \ a, \ abq^{1-n}/ef \ ; \\ aq^{1-n}/e, \ aq^{1-n}/f \ ; \end{matrix} \ q, q^{1-b} \right], $$

which with the help of (4.5.1), gives required result (4.5.3).

To prove (4.5.4), applying the transformation (4.5.2) on the R.H.S. of (4.5.1), it follows that

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, \ a, \ b \ ; \\ e, \ f \ ; \end{matrix} \ q, q^{e+f+n-a-b} \right] $$

$$ = q^{(f-b)n} \frac{(e/a;q)_n \ (abq^{1-n}/ef)_n}{(e;q)_n \ (aq^{1-n}/e)_n} \ {}_3\phi_2\left[\begin{matrix} q^{-n}, f/a, \ f/b \ ; \\ f, \ ef/ab \ ; \end{matrix} \ q, q^{n+e} \right], $$

which on adjusting the parameters gives us (4.5.4).

It is important to note that for $e = 1+a+b-f-n$, (4.5.3) reduces to the following q-analogue of Saalschütz theorem [44, (1.4.3.4), p.28]

$$ (4.5.13) \quad {}_3\phi_2\left[\begin{matrix} q^{-n}, \ a, \ b \ ; \\ e, \ f \ ; \end{matrix} \ q, q \right] = q^{an} \frac{(e/a;q)_n \ (f/a;q)_n}{(e;q)_n \ (f;q)_n} $$

$$= \frac{(f/a;q)_n\ (f/b;q)_n}{(f;q)_n\ (f/ab;q)_n} ,$$

provided $e+f = 1+a+b-n$.

<u>Proof of (4.5.5), (4.5.6) and (4.5.7).</u> To prove (4.5.5) putting $f(u) = \dfrac{\Gamma_q(au,bu)}{\Gamma_q(eu,fu)}\, q^{(e+f-a-b)}$ and $g(u) = \dfrac{\Gamma_q(cu)}{\Gamma_q(gu)}\, q^{(g-c)u}$ in (4.2.11), we have

$$(q^u \Delta_u)^n \left[\frac{\Gamma_q(au,\ bu,\ cu)}{\Gamma_q(eu,\ fu,\ gu)}\, q^{(e+f+g-a-b-c)u} \right]$$

$$= \sum_{r=0}^{n} \left[\begin{matrix} n \\ r \end{matrix} \right] (q^u \Delta_u)^{n-r} \left[\frac{\Gamma_q(auq^r,\ buq^r)}{\Gamma_q(euq^r,\ fuq^r)}\, q^{(e+f-a-b)(u+r)} \right]$$

$$\times\ (q^n \Delta_u)^r \left[\frac{\Gamma_q(eu)}{\Gamma_q(gu)}\, q^{(g-c)u} \right]$$

which on using (4.2.16), (4.2.19) and taking the limit u tending to 0, gives us

$$\lim_{u \to 0} \left[(q^n \Delta_u)^n \frac{\Gamma_q(au,\ bu,\ cu\ )}{\Gamma_q(eu,\ fu,\ gu)}\, q^{(e+f+g-a-b-c)u} \right]$$

$$= \sum_{r=0}^{n} (-)^r q^{-r(r-1-2n)/2}\ \frac{(q^{-n};q)_r}{(q;q)_r}\ (-)^{n-r}\ \frac{\Gamma_q(aq^r,\ bq^r)}{\Gamma_q(eq^r,\ fq^r)}$$

$$\times\ {}_3\phi_2\left[\begin{matrix} q^{-n+r}, aq^r, bq^r ; \\ eq^r, fq^r ; \end{matrix} q, q^{e+f+n-r-a-b}\right] (-)^r \frac{\Gamma_q(c)\,(g/c;q)_r}{\Gamma_q(g)\,(g;q)_r} q^{(e+f-a-b)r},$$

or

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c ; \\ e, f, g ; \end{matrix} q, q^{e+f+g+n-a-b}\right]$$

$$= \sum_{r=0}^{n} (-)^r q^{-r(r-1)/2+(n+e+f-a-b)r} \frac{(q^{-n};q)_r\,(a;q)_r\,(b;q)_r}{(q;q)_r\,(e,q)_r\,(f;q)_r}$$

$$\times\ \frac{(g/c;q)_r}{(g;q)_r}\ {}_3\phi_2\left[\begin{matrix} q^{-n+r}, aq^r, bq^r ; \\ eq^r, fq^r ; \end{matrix} q, q^{e+f+n-r-a-b}\right],$$

which on applying the transformation (4.5.3), changes to

$$(4.5.14)\quad {}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c ; \\ e, f, g ; \end{matrix} q, q^{e+f+g+n-a-b-c}\right]$$

$$= \sum_{r=0}^{n} (-)^r q^{-r(r-1)/2+(n+e+f-a-b)r} \frac{(q^{-n};q)_r\,(a;q)_r}{(q;q)_r\,(e;q)_r}$$

$$\times\ \frac{(b;q)_r\,(g/c;q)_r}{(f;q)_r\,(g;q)_r} q^{(a+r)(n-r)} \frac{(e/a;q)_{n-r}\,(f/a;q)_{n-r}}{(eq^r;q)_{n-r}\,(fq^r;q)_{n-r}}$$

$$\times\ {}_3\phi_2\left[\begin{matrix} q^{-n+r}, aq^r, abq^{1-n+r}/ef ; \\ aq^{1-n+r}/e, aq^{1-n+r}/f ; \end{matrix} q, q^{1-b-r}\right]$$

$$= q^{an} \frac{(e/a;q)_n\,(f/a;q)_n}{(e;q)_n\,(f;q)_n} \sum_{r=0}^{n} \sum_{s=0}^{n-r} (-)^r q^{-r(r-3)/2-br}$$

$$\times \frac{(q^{-n};q)_r\ (a;q)_r\ (b;q)_r\ (g/c;q)_r}{(q;q)_r\ (g;q)_r\ (aq^{1-n}/e;q)_r\ (aq^{1-n}/f;q)_r}$$

$$\times \frac{(q^{-n+r};q)_s\ (aq^r;q)_s\ (abq^{1-n+r}/ef;q)_s}{(q,q)_s\ (aq^{1-n+r}/e;q)_s\ (aq^{1-n+r}/f;q)_s}\ q^{(1+b-r)s}$$

$$= q^{an}\ \frac{(e/a;q)_n\ (f/a;q)_n}{(e;q)_n\ (f;q)_n} \sum_{s=0}^{n} \sum_{r=0}^{s} (-)^r q^{-r(r-3)/2-br+(1-b-r)(s-r)}$$

$$\times \frac{(q^{-n};q)_r\ (a;q)_r\ (b;q)_r\ (g/c;q)_r}{(q;q)_r\ (g;q)_r\ (aq^{1-n}/e;q)_r\ (aq^{1-n}/f;q)_r}$$

$$\times \frac{(q^{-n+r};q)_{s-r}\ (aq^r;q)_{s-r}\ (abq^{1-n+r}/ef;q)_{s-r}}{(q;q)_{s-r}\ (aq^{1-n+r}/e;q)_{s-r}\ (aq^{1-n+r}/f;q)_{s-r}}\ .$$

The above relation after adjusting the parameters and the application of (4.5.2), gives us

(4.5.15) $${}_4\phi_3 \left[\begin{matrix} q^{-n},\ a,\ b,\ c\ ; \\ e,\ f,\ g\ ; \end{matrix}\ q, q^{e+f+g+n-a-b-c} \right]$$

$$= q^{an}\ \frac{(e/a;q)_n\ (f/a;q)_n}{(e;q)_n\ (f;q)_n} \sum_{s=0}^{n} \frac{(q^{-n};q)_s\ (a;q)_s}{(q;q)_s\ (aq^{1-n}/e;q)_s}$$

$$\times \frac{(abq^{1-n}/ef;q)_s\ q^{s(1-b)}}{(aq^{1-n}/f;q)_s}\ {}_3\phi_2 \left[\begin{matrix} q^{-s}, g/c,\ b\ ; \\ g,\ abq^{1-n}/ef\ ; \end{matrix}\ q, q \right]$$

Wrong relation

$$=q^{an}\frac{(e/a;q)_n\,(f/a;q)_n}{(e;q)_n\,(f;q)_n}\sum_{s=0}^{n}\frac{(q^{-n};q)_s\,(a;q)_s\,(abq^{1-n}/ef;q)_s}{(q;q)_s\,(aq^{1-n}/e;q)_s\,(aq^{1-n}/f;q)_s}$$

$$\times\frac{(g/b;q)_s}{(g)_s}\,q^s\;{}_3\phi_2\left[\begin{matrix}q^{-s},\ b,\ abcq^{1-n}/efg\ ;\\ abq^{1-n}/ef,\ bq^{1-s}/g\ ;\end{matrix}\ q,q^{1-c}\right].$$

The above equation (4.5.15) for particular value $e+f+g = 1+a+b+c-n$, gives us

$${}_4\phi_3\left[\begin{matrix}q^{-n},\ a,\ b,\ c\ ;\\ e,\ f,\ g\ ;\end{matrix}\ q,q\right]$$

$$= q^{an}\frac{(e/a;q)_n\,(f/a;q)_n}{(e;q)_n\,(f;q)_n}\sum_{s=0}^{n}\frac{(q^{-n};q)_s\,(a;q)_s\,(g/b;q)_s\,(g/c;q)_s\,q^s}{(q,q)_s\,(aq^{1-n}/e;q)_s\,(aq^{1-n}/f;q)_s\,(g;q)_s}$$

$$= q^{an}\frac{(e/a;q)_n\,(f/a;q)_n}{(e;q)_n\,(f;q)_n}\;{}_4\phi_3\left[\begin{matrix}q^{-n},\ a,\ g/b,\ g/c\ ;\\ g,\ aq^{1-n}/e,\ aq^{1-n}/f\ ;\end{matrix}\ q,q\right],$$

which completes the proof of (4.5.5).

On adjusting the parameters in equation (4.5.5), we can easily get required transformations (4.5.6) and (4.5.7).

Proof of (4.5.8) : Let us take

$f(u) = \Gamma_q(au)q^{(b-a)u}/\Gamma_q(bu)$ in (4.2.22), to get

$$(4.5.16)\qquad \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n}\,(q^u\Delta_u)^n\left[\frac{\Gamma_q(au)}{\Gamma_q(bu)}\,q^{(b-a)u}\right]$$

$$= e_q(-zq^u) \sum_{r=0}^{\infty} \frac{z^r}{(q;q)_r} q^{ru+r(r-1)/2} \frac{\Gamma_q(auq^r)}{\Gamma_q(buq^r)} q^{(b-a)(u+r)}$$

$$= \frac{\Gamma_q(au)}{\Gamma_q(bu)} q^{(b-a)u} e_q(-zq^u) \ {}_1\phi_1^* (a;b;q,zq^{b-a+u})$$

Also from (4.2.19), we can write

$$\text{(4.5.17)} \quad \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} (q^u \Delta_u)^n \left[\frac{\Gamma_q(au)}{\Gamma_q(bu)} q^{(b-a)u} \right]$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} (-)^n q^{(n+b-a)u} \frac{\Gamma_q(au) \ (b/a;q)_n}{\Gamma_q(bu) \ (bu;q)_n}$$

$$= \frac{\Gamma_q(au)}{\Gamma_q(bu)} q^{(b-a)u} \ {}_1\phi_1 (b/a;\ bu\ ;\ q,-zq^u).$$

notation

Now equating (4.5.16) and (4.5.17) for $u=0$, we get (4.5.8).

4.6 <u>Certain Generating Expansions</u> : In this section we shall derive a generating function of generalized hypergeometric series and a few general type of expansion theorems, assuming certain given forms of their expansions.

Putting $f(u) = \Gamma_q((a)u)x^u/\Gamma_q((b)u)$ and $z=-t$ in (4.2.22), we obtain

$$\sum_{n=0}^{\infty} \frac{(-t)^n}{(q;q)_n} (q^u \Delta_u)^n \left[\frac{\Gamma_q((a)u)}{\Gamma_q((b)u)} x^u \right]$$

$$= e_q(tq^u) \sum_{r=0}^{\infty} \frac{(-t)^r}{(q;q)_r} q^{ru+r(r-1)/2} \frac{\Gamma_q((a)uq^r)}{\Gamma_q((b)uq^r)} x^{u+r}$$

$$= \frac{\Gamma_q((a)u)}{\Gamma_q((b)u)} x^u e_q(tq^u) \; {}_A\phi_B\,((a)u;\ (b)u;\ q,-txq^u).$$

Also in view of (4.2.16), we get

$$\sum_{n=0}^{\infty} \frac{(-t)^n}{(q;q)_n} (q^u \Delta_u)^n \left[\frac{\Gamma_q((a)u)}{\Gamma_q((b)u)} x^u \right]$$

$$= \frac{\Gamma_q((a)u)}{\Gamma_q((b)u)} x^u \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} q^{-nu} \; {}_{1+A}\phi_B\,(q^{-n},(a)u;(b)u;q,xq^n).$$

Hence, we get the following generating relation

$$(4.6.1) \quad \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \; {}_{1+A}\phi_B\,(q^{-n},(a)\ ;\ (b)\ ;\ q,\ xq^n)$$

$$= e_q(t) \; {}_A\phi_B^{*}\,((a);\ (b);\ q,\ -tx).$$

<u>Theorem I</u> : If

$$(4.6.2) \quad e_{1/q}(t)G(xt) = \sum_{n=0}^{\infty} c_n(x)t^n \ ; \ G(u) = \sum_{n=0}^{\infty} d_n u^n ,$$

then

$$(4.6.3) \quad F(xt) = \sum_{n=0}^{\infty} \frac{((a);q)_n}{((b);q)_n} c_n(x) \; {}_A\phi_B \left[\begin{matrix} (a)q^n\ ; \\ (b)q^n\ ; \end{matrix} \ q,-t \right] t^n ,$$

where

$$(4.6.4)\qquad F(u) = \sum_{n=0}^{\infty} \frac{((a);q)_n}{((b);q)_n} d_n u^n .$$

Proof : Using the result $e_q(a)e_{1/q}(-a) = 1$, equation (4.6.2) can also be written as

$$G(xt) = e_q(-t) \sum_{n=0}^{\infty} c_n(x)\ t^n$$

or

$$\sum_{n=0}^{\infty} d_n (xt)^n = \sum_{n,r=0}^{\infty} \frac{(-)^r c_n(x) t^{n+r}}{(q;q)_r} .$$

Multiplying both sides by $\Gamma_q((a)u)/\Gamma_q((b)u)$ and replacing t by $t\,E_u$, we get

small c

$$\sum_{n=0}^{\infty} d_n (xt)^n E_u^n \left[\frac{\Gamma_q((a)u)}{\Gamma_q((b)u)}\right] = \sum_{n,r=0}^{\infty} (-)^r \frac{\cancel{\phi}_n(x)}{(q;q)_r}\ t^{n+r}\ E_u^{n+r}\left[\frac{\Gamma_q((a)u)}{\Gamma_q((b)u)}\right].$$

This yields

$$\sum_{n=0}^{\infty} \frac{((a)u;q)_n}{((b)u;q)_n} d_n (xt)^n = \sum_{n=0}^{\infty} \frac{((a)u;q)_n}{((b)u;q)_n} \cancel{\phi}_n(x)$$

small c

$$\times\ {}_A\phi_B\ ((a)uq^n;\ (b)uq^n;\ q,-t)\ t^n ,$$

which on putting $u=0$, gives the required result (4.6.3).

In particular if we take $A=1$, $B=0$, $a_1=a$ and q tending to 1 in the above theorem, we get the following known result due to Rainville [85, Theo.46] (after some modification).

If $e^t G(xt) = \sum_{n=0}^{\infty} c_n(x) t^n$; $G(u) = \sum_{n=0}^{\infty} d_n u^n$,

then for arbitrary a ,

$$(1-t)^{-a} F[xt / (1-t)] = \sum_{n=0}^{\infty} (a)_n \, c_n(x) \, t^n ,$$

where $F(u) = \sum_{n=0}^{\infty} (a)_n \, d_n \, u^n$.

<u>Theorem II</u> : If

$$(4.6.5) \quad E(q,t) \, G(xt) = \sum_{n=0}^{\infty} c_n(x) t^n \; ; \; G(u) = \sum_{n=0}^{\infty} d_n \, u^n ,$$

then

$$(4.6.6) \quad E(q,t) \, F(xyt) = \sum_{n,r=0}^{\infty} \frac{((a);q)_n \, c_n(x)}{((b);q)_n \, (q;q)_n} \, y^n \, t^{n+r}$$

$$X \, q^{r(r-1)/4+n(n-1)/2} \; {}_{A+1}\phi_B \left[\begin{matrix} q^{-r}, \; (a)q^n \; ; \\ (b)q^n \; ; \end{matrix} \; q, yq^{n+r} \right]$$

where

$$(4.6.7) \quad F(u) = \sum_{n=0}^{\infty} d_n \frac{((a);q)_n}{((b);q)_n} \, q^{n(n-1)/2} \, u^n .$$

<u>Proof</u>. Multiply both sides of (4.6.5), by $E(q,-t) \, \Gamma_q((a)u) y^u / \Gamma_q((b)u)$ and replace t by $tq^u E_u$, to get

$$\sum_{n=0}^{\infty} d_n (xt)^n \, (q^u E_u)^u \left[\frac{\Gamma_q((a)u)}{\Gamma_q((b)u)} \, y^u \right]$$

$$= \sum_{n=0}^{\infty} c_n(x)E(q,-tq^u(1+\Delta_u))t^n(q^uE_u)^n \left[\frac{\Gamma_q((a)u)}{\Gamma_q((b)u)} y^u \right]$$

which on using the formula, $E_u f(u) = f(u+1)$ changes to

$$\sum_{n=0}^{\infty} d_n(xt)^n q^{n(n-1)/2+nu} \frac{\Gamma_q((a)uq^n)}{\Gamma_q((b)uq^n)} y^{u+n}$$

Cf C

$$= E(q,-tq^u) \sum_{n,r=0}^{\infty} (-)^r \frac{\cancel{c}_n(x)}{(q;q)_r} q^{r(r-1)/4+n(n-1)/2} t^{n+r}$$

$$X\ (q^u \Delta_u)^r \left[\frac{\Gamma_q((a)uq^n)}{\Gamma_q((b)uq^n)} y^{n+u} q^{nu} \right].$$

Now on the R.H.S. applying the result (4.2.16) and setting u=0, we get (after adjustment of parameters)

$$\sum_{n=0}^{\infty} d_n(xyt)^n q^{n(n-1)/2} \frac{((a);q)_n}{((b);q)_n}$$

Cf C

$$= E(q,-t) \sum_{n,r=0}^{\infty} \frac{\cancel{c}_n(x)\ ((a);q)_n}{(q;q)_n\ ((b);q)_n} q^{r(r-1)/4+n(n-1)/2} y^n$$

$$X\ {}_{A+1}\phi_B\ (q^{-r},(a)q^n;(b)q^n;q,yq^{n+r})t^{n+r}\ .$$

Hence theorem II is proved.

Lastly, by utilising the result

$$e_{1/q}(x) = \sum_{r=0}^{\infty} x^r q^{r(r-1)/2} / (q,q)_r$$

in equation (4.6.2), it follows that

$$\sum_{n=0}^{\infty} c_n(x)\ t^n\ (q^u \Delta_u)^n \left[\frac{\Gamma_q((a)u)}{\Gamma_q((b)u)}\ y^u \right]$$

$$= \sum_{n,\,r=0}^{\infty} d_n\ \frac{x^n\ t^{n+r}\ q^{r(r-1)/2}}{(q;q)_r}\ (q^u \Delta_u)^{n+r} \left[\frac{\Gamma_q((a)u)}{\Gamma_q((b)u)}\ y^u \right],$$

which leads to the following theorem :

<u>Theorem III</u>. If

$$e_{1/q}(t)\ G(xt) = \sum_{n=0}^{\infty} c_n(x) t^n\ ;\ G(u) = \sum_{n=0}^{\infty} d_n\ u^n ,$$

then

$$(4.6.8)\quad \sum_{n=0}^{\infty} c_n(x) t^n\ {}_{A+1}\phi_B\ (q^{-n},(a);(b);q,yq^n)$$

$$= \sum_{n,\,r=0}^{\infty} d_n\ \frac{x^n\ q^{r(r-1)/2}}{(q;q)_r}\ {}_{A+1}\phi_B\ (q^{-n-r},(a);(b);q,\ yq^{n+r}) .$$

4.7 <u>Applications</u> : Multiplying both sides of the identity

$$\frac{\Gamma_q(auq,bu)}{\Gamma_q(eu,\ fu)} - \frac{\Gamma_q(au,buq)}{\Gamma_q(eu,\ fu)} = \frac{q^u(q^b-q^a)}{(1-q)} \cdot \frac{\Gamma_q(au,\ bu)}{\Gamma_q(eu,\ fu)} ,$$

by $x^u q^{-nu}$ and operating with $q^u \Delta_u$, n times, we obtain (after using the result (4.2.16)

$$(-)^n \frac{\Gamma_q(auq,bu)}{\Gamma_q(eu,\ fu)} x^u \ {}_3\phi_2 \left[\begin{matrix} q^{-n}, & auq, & bu\ ; \\ & eu, & fu\ ; \end{matrix} \ q,x \right]$$

$$+(-)^n \frac{\Gamma_q(au,buq)}{\Gamma_q(eu,\ fu)} x^u \ {}_3\phi_2 \left[\begin{matrix} q^{-n}, & au, & buq\ ; \\ & eu, & fu\ ; \end{matrix} \ q,\ x \right]$$ molation

$$= \frac{(q^b-q^a)}{(1-q)} (-)^n \frac{\Gamma_q(au,bu)}{\Gamma_q(eu,fu)} x^u q^u \ {}_3\phi_2 \left[\begin{matrix} q^{-n}, & au, & bu\ ; \\ & eu, & fu\ ; \end{matrix} \ q,qx \right].$$

Now putting $u=0$, and adjusting the parameters, we get the following three term relation

$$(4.7.1) \quad (1-q^a)\ {}_3\phi_2 \left[\begin{matrix} q^{-n}, & aq, & b\ ; \\ & e, & f\ ; \end{matrix} \ q,x \right] - (1-q^b)\ {}_3\phi_2 \left[\begin{matrix} q^{-n}, & a, & bq\ ; \\ & e, & f\ ; \end{matrix} \ q,x \right]$$

$$= (q^b-q^a)\ {}_3\phi_2 \left[\begin{matrix} q^{-n}, & a, & b\ ; \\ & e, & f\ ; \end{matrix} \ q,\ xq \right].$$

The substitution $x=q$, $b = f$ in equation (4.7.1) and the use of the transformation (4.5.2), gives another three term relation

molation

$$(4.7.2) \quad \frac{(fq^{-1}/a;q)_n}{(f;q)_n} (1-q^a) q^{(a+1)n} \ {}_3\phi_2 \left[\begin{matrix} q^{-n}, & aq, & e/f\ ; \\ & e, & aq^{2-n}/f\ ; \end{matrix} \ q,q \right]$$

$$- \frac{(f/a;q)_n}{(f;q)_n} (1-q^f) q^{an} \cdot \quad {}_3\phi_2 \left[\begin{matrix} q^{-n}, \ a, \ eq^{-1}/f \ ; \\ e, \ aq^{1-n}/f \ ; \end{matrix} \ q, q^2 \right]$$

$$= (q^f - q^a) \ {}_2\phi_1 \ (q^{-n}, \ a \ ; \ e \ ; \ q, \ q^2 \ ),$$

which with the help of Saalschütz theorem (4.5.13), yields the following interesting result

Notation

$$(4.7.3) \quad \frac{(f/a;q)_n}{(f;q)_n} (1-q^f) \ {}_3\phi_2 \left[\begin{matrix} q^{-n}, \ a, \ eq^{-1}/f \ ; \\ e, \ aq^{1-n}/f \ ; \end{matrix} \ q, q^2 \right]$$

$$+ q^{-an} (q^f - q^a) \ {}_2\phi_1 \left[\begin{matrix} q^{-n}, \ a \ ; \\ e \ ; \end{matrix} \ q, q^2 \right] = q^n (1-q^a) \ \frac{(eq^{-n}/a;q)_n}{(e;q)_n} \ .$$

Again, in the relation (4.7.1) taking $x = q^{n+e+f-a-b-1}$, $e = 1+a+b-f-n$ and making the use of the transformation (4.5.4), we get

$$(1-q^a) \ \frac{(bq^{-n}/f;q)_n \ (q;q)_n \ q^{(f-b)n}}{(fq/b;q)_n \ (abq^{1-n}/f;q)_n} \ {}_3\phi_2 \left[\begin{matrix} \cancel{q^{-n}}, f/b, fq^{-1}/a; \\ f \ , \ \cancel{q^{-n}} \ ; \end{matrix} \ q, q^{1+a+b-f} \right]$$

$$- (1-q^b) \ \frac{(bq^{1-n}/f;q)_n \ (q;q)_n \ q^{(f-b-1)n}}{(f/b;q)_n \ (abq^{1-n}/f;q)_n} \ {}_3\phi_2 \left[\begin{matrix} \cancel{q^{-n}}, fq^{-1}/b, f/a; \\ f \ , \ \cancel{q^{-n}}; \end{matrix} \ q, q^{1+a+b-f} \right]$$

notation

$$= (q^b - q^a) \ {}_3\phi_2 \left[\begin{matrix} q^{-n}, \ a, \ b \ ; \\ f, \ abq^{1-n}/f \ ; \end{matrix} \ q;q \right] ..$$

Now applying the Saalschutz's theorem (4.5.13) on the R.H.S. and simplifying it, we get the following q-analogue of a result due to Agrawal and Manglik [4, (4.1)]

$$(4.7.4) \quad (1-q^a)\ {}_2\phi_1\ (f/b,\ fq^{-1}/a;\ f;\ q, q^{1+a+b-f})_n$$

$$- (1-q^b)\ {}_2\phi_1\ (fq^{-1}/b, f/a;\ f\ ;\ q, q^{1+a+b-f})_n$$

$$= q^{(1+a+b-f)n}\ (q^b-q^a)\ \frac{(f/a;q)_n\ (f/b;q)_n}{(q;q)_n\ (f;q)_n}\ ,$$

where the suffix n indicates that only first n terms of the series have been taken.

Similarly, if we start with the identities

$$\frac{\Gamma_q(auq, bu)}{\Gamma_q(euq, fu)} - \frac{\Gamma_q(au, bu)}{\Gamma_q(eu, fu)} = q^u\ \frac{(q^e-q^a)}{(1-q)}\ .\ \frac{\Gamma_q(au, bu)}{\Gamma_q(equ, fu)}$$

and

$$\frac{\Gamma_q(au, bu)}{\Gamma_q(equ, fu)} - \frac{\Gamma_q(au, bu)}{\Gamma_q(eu, fqu)} = q^u\ \frac{(q^e-q^f)}{(1-q)}\ .\ \frac{\Gamma_q(au, bu)}{\Gamma_q(equ, fqu)}\ ,$$

and proceed as above we get the following four relations :

$$(4.7.5) \quad (1-q^a)\ {}_3\phi_2\left[\begin{matrix} q^{-n}, aq, b\ ; \\ eq, f\ ; \end{matrix}\ q, x\right] - (1-q^e)\ {}_3\phi_2\left[\begin{matrix} q^{-n}, a, b\ ; \\ e, f\ ; \end{matrix}\ q, x\right]$$

$$= (q^{e}-q^{a}) \ {}_3\phi_2 \begin{bmatrix} q^{-n}, a, b ; & \\ & q, xq \\ eq, f ; & \end{bmatrix},$$

(4.7.6) $$(1-q^{a}) \ {}_2\phi_1 \ (f/b, fq^{-1}/a; f ; q, q^{1+a+b-f})_n$$

$$+ q^{a+b-f} \ (1-q^{f-a-b}) \ {}_2\phi_1 \ (f/b, f/a ; f ; q, q^{a+b-f})_n$$

$$= q^{a+(1+a+b-f)n} \ (q^{b-f-n}-1) \ \frac{(f/a;q)_n \ (f/b;q)_n}{(q;q)_n \ (f;q)_n},$$

(4.7.7) $$(1-q^{f}) \ {}_3\phi_2 \begin{bmatrix} q^{-n}, a, b ; & \\ & q, x \\ eq, f ; & \end{bmatrix} - (1-q^{e}) \ {}_3\phi_2 \begin{bmatrix} q^{-n}, a, b ; & \\ & q, x \\ e, fq; & \end{bmatrix}$$

$$= (q^{e}-q^{f}) \ {}_3\phi_2 \begin{bmatrix} q^{-n}, a, b ; & \\ & q, qx \\ eq, fq ; & \end{bmatrix}$$

and

(4.7.8) $$(1-q^{f}) \ {}_2\phi_1 \ (f/b, f/a ; f ; q, q^{a+b-f})_n$$

$$- (1-q^{a+b-f-1}) \ {}_2\phi_1 \ (fq/b, fq/a ; fq ; q, q^{a+b-f-1})_n$$

$$= q^{(a+b-f-1)(n+1)} \ (1-q^{2f-a-b+n+1}) \ \frac{(fq/a;q)_n \ (fq/b;q)_n}{(q;q)_n \ (fq;q)_n}.$$

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CHAPTER - V

SOME TRANSFORMATIONS, SUMS AND EXPANSIONS INVOLVING BASIC HYPERGEOMETRIC SERIES OF TWO VARIABLES.

5.1 Introduction : There exists a considerable literature on the subject of transformations, summations and expansions of ordinary hypergeometric series of two variables. But the literature in basic multiple hypergeometric series seems to be a lot less extensive.

In this Chapter, through operational technique, some transformations, summation formulas, generating and finite expansions have been derived for basic hypergeometric series of two variables. First we have given a Rodrigues' type representation of the basic hypergeometric series of two variables and then have used it to find out various formulae. Some of the results thus obtained provide, the q-analogues of the known results which otherwise are not easily derivable and others are believed to be new.

We define the basic hypergeometric functions of two variables:

$$(5.1.1) \quad \Phi\left[\begin{matrix}(a) : (b) ; (c) ; \\ (f) : (g) ; (h) ;\end{matrix} \; q ; x , y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{((a);q)_{m+n} \; ((b);q)_m \; ((c);q)_n}{((f);q)_{m+n} \; ((g);q)_m \; ((h);q)_n} \cdot \frac{x^m \; y^n}{(q;q)_m \; (q;q)_n}$$

and

$$(5.1.2) \quad \phi^* \left[\begin{matrix} (a) & : & (b) & ; & (c) & ; \\ (f) & : & (g) & ; & (h) & ; \end{matrix} \; q \; ; \; x \; , \; y \right]$$

$$= \sum_{m,n=0}^{\infty} \frac{((a);q)_{m+n} \; ((b);q)_m \; ((c);q)_n}{((f);q)_{m+n} \; ((g);q)_m \; ((h);q)_n} \cdot \frac{x^m \; y^n}{(q;q)_m (q;q)_n} \; q^{m(m-1)/2} .$$

and the definitions and notations are those, given in the previous chapter.

5.2 <u>A Rodrigues' Type Formula For Basic Hypergeometric Series of Two Variables</u> : From (4.2.10), we have

$$(q^u \triangle_u)^m \; (q^v \triangle_v)^n \; \left[\frac{\Gamma_q \; ((a)uv, \; (b)u, \; (c)v)}{\Gamma_q \; ((f)uv, \; (g)u, \; (h)v)} \; q^{-(mu+nv)} \, x^u \; y^v \right]$$

$$= \sum_{r=0}^{m} \sum_{s=0}^{n} (-)^{m+n-r-s} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{r(r-1)/2+s(s-1)/2-mr-ms}$$

$$\times \frac{\Gamma_q \; ((a)uvq^{r+s}, \; (b)uq^r, \; (c)vq^s)}{\Gamma_q \; ((f)uvq^{r+s}, \; (g)uq^r, \; (h)vq^s)} \; x^{u+r} \; y^{v+s}$$

$$= (-)^{m+n} \; \frac{\Gamma_q \; ((a)uv \; , \; (b)u, \; (c)v)}{\Gamma_q \; ((f)uv, \; (g)u, \; (h)v)} \; x^u \; y^v$$

$$X \sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(q^{-m};q)_r \ (q^{-n};q)_s \ ((a)uv;q)_{r+s} \ ((b)u;q)_r \ ((c)v,q)_s}{(q;q)_r \ (q;q)_s \ ((f)uv;q)_{r+s} \ ((g)u;q)_r \ ((h)v;q)_s} x^r y^s .$$

Therefore

$$(5.2.1) \quad \phi \left[\begin{matrix} (a)uv : & q^{-m}, & (b)u;q^{-n}, & (c)v ; \\ (f)uv : & & (g)u; & (h)v ; \end{matrix} \ q;\ x,\ y \right]$$

$$= (-)^{m+n} \frac{\Gamma_q \ ((f)uv,\ (g)u,\ (h)v)}{\Gamma_q \ ((a)uv,\ (b)u,\ (c)v)} x^{-u} y^{-v}$$

$$X \ (q^u \triangle_u)^m \ (q^v \triangle_v)^n \left[\frac{\Gamma_q \ ((a)uv,\ (b)u,\ (c)v)}{\Gamma_q \ ((f)uv,\ (g)u,\ (h)v)} q^{-(mu+nv)} \ x^u \ y^v \right]$$

which is the q-analogue of the result already given by Agrawal [7].

5.3 <u>Certain Transformations</u> : In this section by using the Rodrigues' type formula (5.2.1) we shall derive certain transformations of basic hypergeometric series of two variables mentioned below and also discuss some of their special cases :

$$(5.3.1) \quad \phi \left[\begin{matrix} a : & q^{-m}, & b ; & q^{-n}, & (c) ; \\ f : & & g ; & & (h) ; \end{matrix} \ q;q^{m+f+g-a-b}, \ y \right]$$

$$= \frac{(g/b;q)_m}{(g;q)_m} \ \phi \left[\begin{matrix} - : & q^{-m}, & b, & f/a ; & q^{-n}, & a, & (c) ; \\ f : & & bq^{1-m}/g ; & & & & (h) ; \end{matrix} \ q;q,y \right] .$$

$$(5.3.2)\quad \phi\left[\begin{matrix} a : q^{-m},\ b ;\ q^{-n},\ (c)\ ; \\ f :\quad g ;\quad (h)\ ; \end{matrix}\ q;q^{m+f+g-a-b},\ y\right]$$

$$= \frac{(f/a;q)_m}{(f;q)_m}\ \phi\left[\begin{matrix} a : q^{-m},\ g/b ;\ q^{-n},\ (c)\ ; \\ - : g, aq^{1-m}/f;\ fq^m,\ (h)\ ; \end{matrix}\ q;q,y\right]$$

$$(5.3.3)\quad \phi\left[\begin{matrix} - : q^{-m}, b, f/a\ ;\ q^{-n}, a, (c)\ ; \\ f :\quad bq^{1-m}/g;\quad (h)\ ; \end{matrix}\ q;q,y\right]$$

$$= \frac{(g;q)_m\ (f/a;q)_m}{(f;q)_m\ (g/b;q)_m}\ \phi\left[\begin{matrix} a : q^{-m},\ g/b ;\ q^{-n}, (c)\ ; \\ - : g, aq^{1-m}/f;\ fq^m, (h)\ ; \end{matrix}\ q;q,y\right].$$

<u>Proof</u>. From (4.2.16), we have

$$(5.3.4)\quad (q^v \Delta_v)^n [(q^u \Delta_u)^m \left\{ \frac{\Gamma_q(auv,\ bu)}{\Gamma_q(fuv,\ gu)}\ q^{(f+g-a-b)u} \right\} \frac{\Gamma_q((c)v)}{\Gamma_q((h)v)}\ q^{-nv} y^v]$$

$$= (-)^m\ \frac{\Gamma_q(bu)}{\Gamma_q(gu)}\ q^{(m+f+g-a-b)u}\ (q^v\Delta_v)^n\ [\frac{\Gamma_q(auv,\ (c)v)}{\Gamma_q(fuv, (h)v)}\ q^{-nv}\ y^v$$

$$\times\ {}_3\phi_2\left[\begin{matrix} q^{-m},\ auv,\ bu\ ; \\ fuv,\ gu\ ; \end{matrix}\ q, q^{m+f+g-a-b}\right]].$$

The application of the transformation (4.5.1) in the above result (5.3.4), gives us

$$(5.3.5)\quad (q^u\Delta_u)^m\ (q^v\Delta_v)^n\ [\frac{\Gamma_q(auv,\ bu,\ (c)v)}{\Gamma_q(fuv,\ gu, (h)v)}\ q^{(f+g-a-b)u-nv}\ y^v]$$

$$= (-)^{m} \frac{\Gamma_q(bu)}{\Gamma_q(gu)} q^{(m+f+g-a-b)u} (q^{v}\Delta_{v})^{n} \left[\frac{\Gamma_q(auv, (c)v)}{\Gamma_q(fuv, (h)v)}\right.$$

$$\left. X\ q^{-nv} y^{v} \frac{(g/b;q)_m}{(gu;q)_m} {}_3\phi_2 \left[\begin{matrix} q^{-m}, bu, f/a ; \\ fuv, bq^{1-m}/g ; \end{matrix} q,q \right] \right]$$

$$= (-)^{m+n} \frac{\Gamma_q(bu)\ (g/b;q)_m}{\Gamma_q(gu)\ (gu;q)_m} q^{(m+f+g-a-b)u} \sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(q^{-m};q)_r}{(q;q)_r} \frac{(q^{-n};q)_s}{(q;q)_s}$$

$$X \frac{\Gamma_q(auvq^{s}, (c)vq^{s})\ (bu;q)_r\ (f/a;q)_r}{\Gamma_q(fuvq^{s}, (h)vq^{s})\ (fuvq^{s};q)_r\ (bq^{1-m}/g;q)_r} q^{r} y^{v+s}$$

$$= (-)^{m+n} \frac{\Gamma_q(auv, bu, (c)v)}{\Gamma_q(fuv, gu, (h)v)} q^{(m+f+g-a-b)u} y^{v}$$

$$X \frac{(g/b;q)_m}{(gu;q)_m} \phi \left[\begin{matrix} - : q^{-m}, bu, f/a ; q^{-n}, auv, (c)v ; \\ fuv: \quad bq^{1-m}/g ; \quad (h)v ; \end{matrix} q;q,y \right].$$

Next on replacing a,b,e and f respectively by a+u+v, b+u, f+u+v and g+u in the transformation (4.5.1), we have

$${}_3\phi_2 \left[\begin{matrix} q^{-m}, auv, bu ; \\ fuv, gu ; \end{matrix} q,q^{m+f+g-a-b} \right]$$

$$= \frac{(f/a;q)_m}{(fuv;q)_m} {}_3\phi_2 \left[\begin{matrix} q^{-m}, auv, g/b ; \\ gu, aq^{1-n}/f ; \end{matrix} q,q \right].$$

Thus equation (5.3.4) can be written as

$$(5.3.6)\quad (q^u\Delta_u)^m\ (q^v\Delta_v)^n\ \left[\frac{\Gamma_q(auv,\ bu,(c)v)}{\Gamma_q(fuv,\ gu,(h)v)}\, q^{(f+g-a-b)u-nv}\ y^v\right]$$

$$= (q^v\ \Delta_v)^n\ \left[\ (-)^m\ \frac{\Gamma_q(auv,\ bu)}{\Gamma_q(fuv,\ gu)}\ q^{(m+f+g-a-b)u}\right.$$

$$\left. \times\ \frac{(f/a;q)_m}{(fuv;q)_m}\ {}_3\phi_2\left[\begin{matrix} q^{-m},\ auv,\ g/b\ ; \\ gu,\ aq^{1-m}/f\ \ ; \end{matrix}\ q,q\right]\frac{\Gamma_q((c)v)}{\Gamma_q((h)v)}\ q^{-nv}\ y^v\ \right]$$

$$= (-)^{m+n}\ q^{(m+f+g-a-b)u}\ \frac{\Gamma_q(bu)\ \ (f/a;q)_m}{\Gamma_q\ \ (gu)}\sum_{r=0}^{m}\ \sum_{s=0}^{n}\frac{(q^{-m};q)_r\ \ (q^{-n};q)_s}{(q;q)_r\ \ (q;q)_s}$$

$$\times\ \frac{\Gamma_q(auvq^s,\ \ (c)vq^s)\ \ (auvq^s;q)_r\ \ (g/b;q)_r}{\Gamma_q(fuvq^{s+m},(h)vq^s)\ \ (gu;q)_r\ \ (aq^{1-m}/f;q)_r}\ q^r\ y^{v+s}$$

$$= (-)^{m+n}\ q^{(m+f+g-a-b)u}\ y^v\ \ \frac{\Gamma_q(auv,\ bu,\ (c)v)}{\Gamma_q(fuv,\ gu,\ (h)v)}$$

$$\times\ \frac{(f/a;q)_m}{(fuv;q)_m}\ \phi\left[\begin{matrix} auv\ :\ q^{-m},\ g/b;\ q^{-n},\ \ (c)v\ \ \ \ \ \ \ \ ; \\ -\ :\ gu\ \ ,\ aq^{1-m}/f\ ;\ fuvq^m,(h)v\ ; \end{matrix}\ q;q,y\right].$$

But from the Rodrigues' type formula (5.2.1), we see that

$$(5.3.7)\quad (q^u\Delta_u)^m\ (q^v\Delta_v)^n\ \left[\frac{\Gamma_q(auv,\ bu,(c)v)}{\Gamma_q(fuv,\ gu,(h)v)}\, q^{(f+g-a-b)u-nv}\ y^v\right]$$

$$= (-)^{m+n}\ \frac{\Gamma_q(auv,\ bu,\ \ (c)v)}{\Gamma_q(fuv,\ gu,\ \ (h)v)}\ q^{(m+f+g-a-b)u}\ y^v$$

$$\times\ \phi\left[\begin{matrix} auv : q^{-m}, bu\ ;\ q^{-n},\ (c)v\ ; \\ fuv :\qquad gu\ ;\qquad (h)v\ ; \end{matrix}\ q; q^{m+f+g-a-b},\ y\right].$$

Hence on equating (5.3.5), (5.3.6) and (5.3.7), we get the required results (5.3.1), (5.3.2) and (5.3.3).

Special Cases : (i) If we take $C=H=1$, $c_1=c$, $h_1=h$ and replace g by $1+b-g-m$ in (5.3.1), we have

$$(5.3.8)\quad \phi\left[\begin{matrix} - : q^{-m},\ b,\ f/a\ ;\ q^{-n},\ a,\ c\ ; \\ f :\qquad\qquad g\ ;\qquad\qquad h\ ; \end{matrix}\ q; q, q^{n+f+h-a-c}\right]$$

$$= q^{bm}\,\frac{(g/b;q)_m}{(g;q)_m}\,\phi\left[\begin{matrix} a : q^{-m},\ b\ ;\ q^{-n},\ c\ ; \\ f : bq^{1-m}/g;\qquad h\ ; \end{matrix}\ q; q^{1+f-a-g},\ q^{n+f+h-a-c}\right],$$

which is the q-analogue of a result due to Singal [101,(2.2)].

Again applying the transformation (5.3.1) on the R.H.S. of (5.3.8), we get the following q-analogue of another result due to Singal [100,(1.3)].

$$(5.3.9)\quad \phi\left[\begin{matrix} - : q^{-m},\ b, f/a\ ;\ q^{-n}, a, c\ ; \\ f :\qquad g\ ;\qquad h\ ; \end{matrix}\ q; q, q^{n+f+h-a-c}\right]$$

$$= q^{bm}\,\frac{(g/b;q)_m\ (h/c;q)_n}{(g;q)_m\ \ (h;q)_n}\,\phi\left[\begin{matrix} - : q^{-m},\ a, b\ ;\ q^{-n}, c, f/a\ ; \\ f : bq^{1-m}/g\ ;\ cq^{1-n}/h\ ; \end{matrix}\ q; q^{1+f-a-g}, q\right].$$

(ii) If we let $C=H=1$, $c_1=c$, $h_1=h$, $y=q^{1+n+h-m-f-c}$ and f replaced by $1+a-m-f$ in (5.3.2), we observe that

$$(5.3.10)\quad \phi\left[\begin{matrix} a : q^{-m}, g/b ; q^{-n}, c ; \\ - : g, f \quad ; aq/f, h ; \end{matrix} q;q,q^{1+n+h-m-f-c}\right]$$

$$= q^{am}\frac{(f/a;q)_m}{(f;q)_m}\,\phi\left[\begin{matrix} a \quad : q^{-m}, b ; q^{-n}, c ; \\ aq^{1-m}/f : \quad g ; \quad h ; \end{matrix} q;q^{1-f+g-b}, q^{1+n+h-m-f-c}\right].$$

Again using the transformation (5.3.2) on the R.H.S. of (5.3.10), we get another interesting transformation

$$(5.3.11)\quad \phi\left[\begin{matrix} a : q^{-m}, g/b ; q^{-n}, c ; \\ - : g , f \quad ; aq/f, h ; \end{matrix} q;q,q^{1+n+h-m-f-c}\right]$$

$$= q^{am}\frac{(f/a;q)_m\,(q^{1-m}/f;q)_n}{(f;q)_m\,(aq^{1-m}/f;q)_n}$$

$$\times\ \phi\left[\begin{matrix} a : q^{-m}, b ; q^{-n}, h/c ; \\ - : g, aq^{1-m+n}/f ; h, fq^{m-n} ; \end{matrix} q;q^{1-f+g-b}, q\right].$$

5.4 <u>Certain Summation Formulas</u> : In this section we shall obtain the following summation formulas of basic hypergeometric series of two variables. Most of the results thus obtained provide, the q-analogues of the results given by Srivastava and Saran [109] and Singal [98] which otherwise are not easily derivable :

(i) Under the conditions

$$f+g = 1+a+b-m \quad \text{and} \quad f+h = 1+a+c-n$$

?

If

$$\phi \equiv \phi \left[\begin{matrix} a : q^{-m}, b ; q^{-n}, c ; \\ f : \quad g ; \quad h ; \end{matrix} q,q \right]$$

then

$$(5.4.1) \quad \phi = q^{am} \frac{(q;q)_m \, (f/a;q)_m \, (g/a;q)_{m-n} \, (b;q)_n}{(q;q)_{m-n} \, (g;q)_m \, (f;q)_{m+n}} \; ; \text{ for } h=a,$$

?

$$(5.4.2) \quad \phi = 0 \; ; \quad \text{for} \quad h=c \, ,$$

$$(5.4.3) \quad \phi = \frac{(bc/a;q)_{m+n} \, (b;q)_n \, (c;q)_m}{(bc;q)_{m+n} \, (b/a;q)_n \, (c/a)_m} \; ; \text{ for } f = b+c \, ,$$

and

$$(5.4.4) \quad \phi = q^{am} \frac{(f/a;q)_{m+n} \, (g/a;q)_{m-n}}{(f;q)_{m+n} \, (g;q)_{m-n}} \; , \text{ for } \quad g+h=a+1 \, ,$$

(ii)

$$(5.4.5) \quad \phi \left[\begin{matrix} a : q^{-m}, b ; q^{-n}, c , f/b \quad ; \\ f : abq^{1-m}/f ; \; q^m f/b, acq^{1-m-n}/f \; ; \end{matrix} q;q,q \right]$$

$$= \frac{(f/a;q)_{m+n} \, (f/b;q)_m \, (q^m f/c)_n}{(f;q)_{m+n} \, (f/ab;q)_m \, (fq^m/ac)_n}$$

(iii) If we write

$$S \equiv \phi \left[\begin{matrix} - : q^{-m}, a, b ; q^{-n}, f/a, c ; \\ f : abq^{1-m}/f ; \qquad cq^{1-n}/h ; \end{matrix} q;q,q \right]$$

then

$$(5.4.6) \quad S = \frac{(a;q)_n (f/a;q)_m (b;q)_n (f/b;q)_m}{(f;q)_{m+n} (ab/f;q)_n (f/ab;q)_m} ;$$

for $c=f-b$ and $h=a$,

(due to Srivastava [117]) ,

$$(5.4.7) \quad S = \frac{(f/a;q)_{m+n} (f/b;q)_{m+n}}{(q^m;q)_n (f;q)_{m+n} (f/ab;q)_m} ,$$

for $c=f-b$ and $h = m+f-b$,

$$(5.4.8) \quad S = q^{(b-n)m} \frac{(q;q)_m (f/b;q)_m (d;q)_{m+n}}{(d;q)_n (d;q)_m (f;q)_m (bq;q)_m}$$

for $c = f-b$, $a = f+m$ and $1+c-h-n = d$.

Proof : To prove above results, substitute $g = 1+a+b-f-m$ in (5.3.4), we have

$$(q^u \triangle_u)^m (q^v \triangle_v)^n \left[\frac{\Gamma_q(auv, bu, (c)v)\, q^{(1-m)u-nv} y^v}{\Gamma_q(fuv, abuq^{1-m}/f, (h)v)} \right]$$

$$= (-)^m \frac{\Gamma_q(bu)}{\Gamma_q(abuq^{1-m}/f)} q^u (q^v \Delta_v)^n \left[\frac{\Gamma_q(auv,\ (c)v)}{\Gamma_q(fuv,\ (h)v)} q^{-nv} y^v \right.$$

$$\left. \times\ {}_3\phi_2 \left[\begin{matrix} q^{-m}, & auv, & bu & ; \\ fuv, & abuq^{1-m}/f & ; & \end{matrix} q;q \right] \right]$$

$$= (-)^m \frac{\Gamma_q(bu)\ q^u}{\Gamma_q(abuq^{1-m}/f)} (q^v \Delta_v)^n \left[\frac{\Gamma_q(auv,\ (c)v)}{\Gamma_q(fuv,\ (h)v)} q^{-nv} y^v \right.$$

$$\left. \times \frac{(f/a;q)_m\ (fv/b;q)_m}{(fuv;q)_m\ (f/abu;q)_m} \right]$$

$$= (-)^m \frac{\Gamma_q(bu)\ (f/a;q)_m\ q^u}{\Gamma_q(abuq^{1-m}/f)\ (f/abu;q)_m}$$

$$\times (q^v \Delta_v)^n \left[\frac{\Gamma_q(auv,\ fvq^m/b,\ (c)v)}{\Gamma_q(fv/b,\ fuvq^m,\ (h)v)} q^{-nv} y^v \right].$$

Now using (5.2.1), (4.2.15) and putting $u = v = 0$, we get

$$(5.4.9) \quad \phi \left[\begin{matrix} a\ :\ q^{-m}, & b & ;\ q^{-n}, & (c)\ ; \\ f\ :\ abq^{1-m}/f\ ; & & & (h)\ ; \end{matrix} q;q,y \right]$$

$$= \frac{(f/a;q)_m\ (f/b;q)_m}{(f;q)_m\ (f/ab;q)_m}\ {}_{C+3}\phi_{H+2} \left[\begin{matrix} q^{-n}, a, fq^m/b, (c)\ ; \\ f/b,\ fq^m \qquad ; \end{matrix} q,y \right].$$

For particular values $C=H=1$, $c_1=c$, $h_1=h$ and $y = q$, we can write above transformation as

$$\phi\left[\begin{matrix} a & : & q^{-m}, & b\ ; & q^{-n}, & c\ ; \\ f & : & & g\ ; & & h\ ; \end{matrix}\ g;q,q\right]$$

$$= q^{am}\,\frac{(f/a;q)_m\ (g/a;q)_m}{(f;q)_m\ (g;q)_m}\ {}_4\phi_3\left[\begin{matrix} a^{-n}, & a, & c, & fq^m/b\ ; \\ f/b, & fq^m\ , & & h\ ; \end{matrix}\ q,q\right],$$

provided, $f+g = 1+a+b-m$,

which after putting special values and adjusting the parameters gives required summation formulas (5.4.1), (5.4.2), (5.4.3) and (5.4.4)

Next if we substitute $C=H=2$, $c_1= f-b$, $h_1= f+m-b$, $h_2= 1-m-n+a+c-f$ and $c_2= c$ in (5.4.9) and using Saalschütz's analogue, we get summation formula (5.4.5).

Now taking $C=H=1$, $c_1= c$, $h_1= h$ and $y=q^{n+f+h-a-c}$ in (5.4.9) and applying the transformation (5.3.1), we get

$$\phi\left[\begin{matrix} - & : & q^{-m}, & a, & b\ ; & q^{-n}, & f/a, & c\ ; \\ f & : & abq^{1-m}/f & & ; & cq^{1-n}/h\ ; & & \end{matrix}\ q;q,q\right]$$

$$= \frac{(h;q)_n\ (f/a;q)_m\ (f/b;q)_m}{(h/c;q)_n\ (f;q)_m\ (f/ab;q)_m}$$

$$\times\ {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & fq^m/b, & c\ ; \\ f/b, & fq^m, & & h\ ; \end{matrix}\ q,q^{n+f+h-a-c}\right]$$

which for special values reduces to (5.4.6), (5.4.7) and (5.4.8).

5.5 Certain Generating and Finite Expansions : In this section we have obtained certain generating and finite expansions for basic hypergeometric series of two variables.

Substituting $f(u) = (q^v\Delta_v)^m \left[\frac{\Gamma_q((a)uv,(b)u,(c)v)}{\Gamma_q((f)uv,(g)u,(h)v)} x^u y^v \right]$

and replacing z by $-t$ in equation (4.2.22), we get

$$\sum_{n=0}^{\infty} \frac{(-t)^n}{(q;q)_n} (q^u\Delta_u)^n (q^v\Delta_v)^m \left[\frac{\Gamma_q((a)uv,(b)u,(c)v)}{\Gamma_q((f)uv,(g)u,(h)v)} x^u y^v \right]$$

$$= e_q(tq^u) \sum_{r=0}^{\infty} \frac{(-t)^r q^{ru+r(r-1)/2}}{(q;q)_r}$$

$$\times (q^v\Delta_v)^m \left[\frac{\Gamma_q((a)uvq^r,\ (b)uq^r,\ (c)v)}{\Gamma_q((f)uvq^r,\ (g)uq^r,\ (h)v)} x^{u+r} y^v \right]$$

$$= e_q(tq^u) \sum_{r=0}^{\infty} \sum_{s=0}^{m} \frac{(-)^m (q^{-m};q)_s}{(q;q)_r (q;q)_s} (-t)^r q^{mv+ms+ru+r(r-1)/2}$$

$$\times \frac{\Gamma_q((a)uvq^{r+s},\ (b)uq^r,\ (c)vq^s)}{\Gamma_q((f)uvq^{r+s},\ (g)uq^r,\ (h)vq^s)} x^{u+r} y^{v+s}$$

$$= (-)^m \frac{\Gamma_q((a)uv,\ (b)u,\ (c)v)}{\Gamma_q((f)uv,\ (g)u,\ (h)v)} x^u y^v q^{mv} e_q(tq^u)$$

$$\times \phi^* \left[\begin{matrix} (a)uv : (b)u ; q^{-m},\ (c)v ; \\ (f)uv : (g)u ; \qquad (h)v ; \end{matrix} q; -xtq^u,\ yq^m \right].$$

The use of Rodrigues' type formula (5.2.1) on the L.H.S. of it, gives the following generating expansion (after putting $u=v=0$).

$$(5.5.1)\quad \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \phi \left[\begin{matrix} (a) : q^{-n}, & (b) ; q^{-m}, & (c) ; \\ (f) : & (g) ; & (h) ; \end{matrix} q; q^n x, q^m y \right]$$

$$= e_q(t)\, \phi^* \left[\begin{matrix} (a) : & (b) ; q^{-m}, & (c) ; \\ (f) : & (g) ; & (h) ; \end{matrix} q; -xt, yq^m \right].$$

Next in (5.2.1) setting $m=n$, multiplying both sides by $t^n q^{n(n-1)/4}/(q;q)_n$ and performing the summation from $n = 0$ to $n = \infty$, we get

$$\frac{\Gamma_q((a)uv, (b)u, (c)v)}{\Gamma_q((f)uv, (g)u, (h)v)} x^u y^v \sum_{n=0}^{\infty} \frac{t^n q^{(u+v)n+n(n-1)/4}}{(q;q)_n}$$

$$\times \phi \left[\begin{matrix} (a)uv ; q^{-n}, & (b)u ; q^{-n}, & (c)v ; \\ (f)uv ; & (g)u ; & (h)v ; \end{matrix} q; xq^n, yq^n \right]$$

$$= E(q, tq^{u+v} \Delta_u \Delta_v) \left[\frac{\Gamma_q((a)uv, (b)u, (c)v)}{\Gamma_q((f)uv, (g)v, (h)v)} x^u y^v \right]$$

$$= E(q, tq^{u+v})\, E(q, tq^u E_u q^v E_v)\, E(q, -tq^v q^u E_u)$$

$$\times E(q, -tq^u q^v E_v) \left[\frac{\Gamma_q((a)uv, (b)u, (c)v)}{\Gamma_q((f)uv, (g)u, (h)v)} x^u y^v \right]$$

$$= E(q, tq^{u+v}) \sum_{p,r,s=0}^{\infty} \frac{(-)^{r+s}\, t^{p+r+s}\, q^{us+vr+p(p-1)/4+r(r-1)/4+s(s-1)/4}}{(q,q)_p\ (q,q)_r\ (q,q)_s}$$

$$\times\ (q^u\, E_u)^{p+r}\ (q^v\, E_v)^{p+s}\ \left[\frac{\Gamma_q((a)uv,\ (b)u,\ (c)v)}{\Gamma_q((f)uv,\ (g)u,\ (h)v)}\, x^u\, y^v \right],$$

which on using $E_u\ f(u) = f(u+1)$ and putting $u = v = 0$, gives an interesting generating expansion

$$(5.5.2)\quad \sum_{n=0}^{\infty} \frac{t^n\, q^{n(n-1)/4}}{(q;q)_n}\ \phi\left[\begin{matrix} (a) : q^{-n}, & (b)\ ; q^{-n}, & (c)\ ; \\ (f) : & (g)\ ; & (h)\ ; \end{matrix}\ q; xq^n, yq^n \right]$$

$$= E\ (q,t) \sum_{p,r,s=0}^{\infty} \frac{(-)^{r+s}\, q^{5p(p-1)/4+3r(r-1)/4+3s(s-1)/4+p(r+s)}}{(q;q)_p\ (q;q)_r\ (q;q)_s}$$

$$\times\ \frac{((a);q)_{2p+r+s}\ ((b);q)_{p+r}\ ((c),q)_{p+s}}{((f);q)_{2p+r+s}\ ((g);q)_{p+r}\ ((h);q)_{p+s}}$$

For finite expansions, starting from (5.2.1) and applying the operational formula (4.2.11), we obtain

$$\phi\left[\begin{matrix} (a)uv : q^{-m}, (b)u\ ; & q^{-n}, (c)v\ ; \\ (f)uv : \quad (g)u\ ; & (h)v\ ; \end{matrix}\ q; x, y \right] = \frac{\Gamma_q((f)uv, (g)u, (h)v)}{\Gamma_q((a)uv, (b)u, (c)v)}$$

$$\times\ (-)^{m+n}\, x^{-u}\, y^{-v} \sum_{r=0}^{m} \left[\begin{matrix} m \\ r \end{matrix} \right] (q^u \Delta_u)^r \left[\frac{\Gamma_q(eu)}{\Gamma_q(du)}\, q^{(d-e)u} \right]$$

$$\times\ (q^u \Delta_u)^{m-r} (q^v \Delta_v)^n \left[\frac{\Gamma_q((a)uvq^r, duq^r, (b)uq^r, (c)v)}{\Gamma_q((f)uvq^r, euq^r, (g)uq^r, (h)v)}\, x^{u+r}\, y^v\, q^{(d-e)(u+r)} \right],$$

which by use of (4.2.16), (5.2.1) and the substitution $u = v = 0$, yields

$$\phi\left[\begin{matrix}(a) & : & q^{-m}, & (b) & ; & q^{-n}, & (c) & ; \\ (f) & : & & (g) & ; & & (h) & ;\end{matrix}\; q;x,y\right]$$

$$\sum_{r=0}^{m}\left[\begin{matrix}m\\r\end{matrix}\right]\frac{((a);q)_r\,((b);q)_r\,(d;q)_r}{((f);q)_r\,((g);q)_r\,(e;q)_r}\,q^{(e-d-m)r}\,x^r\;{}_2\phi_1\left[\begin{matrix}q^{-n}, & e\;; \\ & d\;;\end{matrix}\; q,q^{d+n-e}\right]$$

$$\times\,\phi\left[\begin{matrix}(a)q^r & : & q^{r-m}, & (b)q^r, & dq^r & ; & q^{-n}, & (c) & ; \\ (f)q^r & : & & (g)q^r, & eq^r & ; & & (h) & ;\end{matrix}\; q;xq^{e-d-r},\,y\right]$$

We know that [44, p.28]

$$_2\phi_1\,(q^{-n}, a\;;\;c\;;\;q,\;q^{n+c-a}) = (c/a;q)_n/(c;q)_n\;.$$

using the above summation formula, we get the expansion

$$(5.5.3)\quad \phi\left[\begin{matrix}(a) & : & q^{-m}, & (b) & ; & q^{-n}, & (c) & ; \\ (f) & : & & (g) & ; & & (h) & ;\end{matrix}\; q;x,y\right]$$

$$= \sum_{r=0}^{m}\left[\begin{matrix}m\\r\end{matrix}\right] q^{(e-d-m)r}\,x^r\;\frac{((a);q)_r\;((b);q)_r\;(d/e;q)_r}{((f);q)_r\;((g);q)_r\;(e;q)_r}$$

$$\times\,\phi\left[\begin{matrix}(a)q^r & : & q^{r-m}, & (b)q^r, & dq^r & ; & q^{-n}, & (c) & ; \\ (f)q^r & : & & (g)q^r, & eq^r & ; & & (h) & ;\end{matrix}\; q;xq^{e-d-r},\,y\right].$$

Again proceeding in a similar way, we can show that

$$\phi\left[\begin{matrix}(a)uv : q^{-m}, & (b)u ; q^{-n}, & (c)v ; \\ (f)uv : & (g)u ; & (h)v ;\end{matrix} \; q; xw, y\right]$$

$$= (-)^{m+n} \frac{\Gamma_q((f)uv, (g)u, (h)v)}{\Gamma_q((a)uv, (b)u, (c)v)} (xw)^{-u} y^{-v} \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix} (q^u \triangle_u)^{m-r} [x^{u+r}]$$

$$\times (q^u \triangle_u)^r (q^v \triangle_v)^n \left[\frac{\Gamma_q((a)uv, (b)u, (c)v)}{\Gamma_q((f)uv, (g)u, (h)v)} y^v z^u q^{-mu-nv} \right].$$

But, we have

$$(q^u \triangle_u)^n [x^u] = (-)^n q^{nu} . x^u \sum_{r=0}^{n} (-)^r \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-1)/2} x^r$$

$$= (-)^n q^{nu} x^u (1-x)^{(n)}$$

where $(1-x)^{(n)} = (1-\cancel{y})(1-q\cancel{y})\dots(1-q^{n-1}\cancel{y})$ [44, p. 128]. x/ x/ - x/

Hence after putting $u = v = 0$, we get another expansion

(5.5.4) $$\phi\left[\begin{matrix}(a) : q^{-m}, & (b) ; q^{-n}, & (c) ; \\ (f) ; & (g) ; & (h) ;\end{matrix} \; q; xw, y\right]$$

$$= (1-x)^{(m)} \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix} \frac{x^r}{(1-q^{m-r}x)^{(r)}} \phi\left[\begin{matrix}(a) : q^{-r}, & (b) ; q^{-n}, & (c) ; \\ (f) : & (g) ; & (h) ;\end{matrix} \; q; q^{r-m} w, y\right].$$

In a similar way we can easily show that

$$(5.5.5)\quad \phi\left[\begin{matrix}(a) : q^{-m}, & (b) ; q^{-n}, & (c) ; \\ (f) : & (g) ; & (h) ;\end{matrix}\; q; x, yw\right].$$

$$\times \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \frac{((a);q)_r ((c);q)_r ((d);q)_r}{((f);q)_r ((h);q)_r ((e);q)_r}\, y^r\, q^{-nr}\; {}_{E+1}\phi_D\left[\begin{matrix} q^{-r}, (e) ; \\ (d) ; \end{matrix}\; q, wq^r\right]$$

$$\times\; \phi\left[\begin{matrix}(a) : q^{-m}, & (b) ; q^{r-n}, & (c), (d) ; \\ (f) : & (g) ; & (h), (e) ;\end{matrix}\; q; x, yq^{-r}\right],$$

$$(5.5.6)\quad \phi\left[\begin{matrix}(a) : q^{-m}, & (b) ; q^{-n}, & (c) ; \\ (f) : & (g) ; & (h) ;\end{matrix}\; q; xq^m, yq^n\right]$$

$$= \sum_{r=0}^{n} \sum_{s=0}^{m} \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} m \\ s \end{bmatrix} \frac{((a);q)_{r+s}\; ((b);q)_s}{((f);q)_{r+s}\; ((g);q)_s}\, q^{s(s-1)/2}\, (-x)^r\, y^s$$

$$\times\; {}_{A+1}\phi_F\left[\begin{matrix} q^{-n+r}, (a)q^{r+s} ; \\ (f)q^{r+s} ; \end{matrix}\; q, yq^{n-r}\right]\; {}_{C+1}\phi_H\left[\begin{matrix} q^{-r}, (c) ; \\ (h) ; \end{matrix}\; q, q^r\right].$$

$$(5.5.7)\quad \phi\left[\begin{matrix}(a) : q^{-d-m}, & (b) ; q^{-n-e}, & (c) ; \\ (f) : & (g) ; & (h) ;\end{matrix}\; q; x, y\right]$$

d & e are [illegible] integers

$$= \sum_{r=0}^{d} \sum_{s=0}^{e} \frac{((a);q)_{r+s}\; ((b);q)_r\; (q^{-d};q)_r\; ((c);q)_s\; (q^{-e};q)_s}{((f);q)_{r+s}\; ((g);q)_r\; ((h);q)_s\; (q;q)_r\; (q;q)_s}\, x^r\, y^s$$

$$\times\; \phi\left[\begin{matrix}(a)q^{r+s} : q^{-m}, & (b)q^r ; q^{-n}, & (c)q^s ; \\ (f)q^{r+s} : & (g)q^r ; & (h)q^s ;\end{matrix}\; q; xq^{-d}, yq^{-e}\right]$$

and

$$(5.5.8) \quad \phi\left[\begin{array}{lllll} (a) : q^{-m}, & (b) ; q^{-n}, & (c) ; & \\ (f) : & (g) ; & (h) ; & q; xz, yw \end{array}\right]$$

$$= (1-x)^{(m)} (1-y)^{(n)} \sum_{r=0}^{m} \sum_{s=0}^{n} \left[{m \atop r}\right] \left[{n \atop s}\right] \frac{x^r \; y^s}{(1-xq^{m-r})^{(r)} \; (1-yq^{n-s})^{(s)}}$$

$$x \; \phi\left[\begin{array}{lllll} (a) : q^{-r}, & (b) ; q^{-s}, & (c) ; & \\ (f) : & (g) ; & (h) ; & q; zq^{r-m}, wq^{s-n} \end{array}\right]$$

Most of the expansions given above are the q-analogues of the results given by Agrawal [7].

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CHAPTER VI

CERTAIN GENERATING FUNCTIONS INVOLVING LAURICELLA FUNCTIONS

6.1 Introduction : Generating functions play a very important role in the study of hypergeometric type of functions and polynomials. From generating functions, various important and useful properties of the sequences, which they generate, can be obtained. As far as multiple generating functions are concerned, sufficient work is not available in the literature and a lot is still required to be done. The main work in this direction has been done by Srivastava [116], Srivastava and Singhal [124], Carlitz [31], Thakare [128, 129], Mandekar and Thakare [65, 66] and Exton [43]. These multilinear generating functions contain, as special cases, a large number of one dimensional results which are very interesting in nature. These results can be expressed as the combination of the elementary functions, for example [43, (6.2.2), p.189]. 86

$$\sum_{m_1,\ldots,m_n=0}^{\infty} \binom{a_1+(b_1+1)m_1}{m_1} \cdots \binom{a_n+(b_n+1)m_n}{m_n} t_1^{m_1} \cdots t_n^{m_n}$$

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$$\times F_A^{(n)} [a: -m_1, \cdots, -m_n ; 1+a_1+b_1m_1, \cdots, 1+a_n+b_nm_n ; x_1, \cdots, x_n]$$

$$= \frac{(1+w_1)^{a_1+1}}{1-b_1w_1} \cdots \frac{(1+w_n)^{a_n+1}}{1-b_nw_n} (1+w_1 x_1+ \ldots + w_n x_n)^{-a} ,$$

where $w_j = t_j (1+w_j)^{b_j+1}$; $1 \leqslant j \leqslant n$.

In this Chapter, our aim is to establish some general type of bilateral generating relations involving Laguerre/ Jacobi polynomials and functions of several variables. We have also derived certain multiple generating functions for the product of Laguerre / Jacobi polynomials and Lauricella functions. The results thus obtained are very general in nature and in particular leads to several known results. Since it is not possible to mention all possible cases, only a few of them have been discussed as illustration.

6.2 Some Bilateral Generating Functions Involving Lauricella Functions and Laguerre Polynomials : In this section we shall derive two bilateral generating functions mentioned below and also discuss some of their special cases :

(6.2.1) $$\sum_{n=0}^{\infty} \frac{(w)_n}{(1+a)_n} L_n^{(a)}(x) \Psi_2^{(r)} [w+n: b_1, \cdots, b_r ; x_1, \cdots, x_r] t^n$$

$$= (1-t)^{-w} \Psi_2^{(r+1)} [w : b_1, \cdots, b_r, 1+a ; \frac{x_1}{1-t}, \cdots, \frac{x_r}{1-t}, \frac{-xt}{1-t}]$$

and

(6.2.2)

$$\sum_{n=0}^{\infty} L_n^{(a)}(x)\, F_A^{(r+1)}\left[w: b_1,\cdots,b_r,-n;\ c_1,\cdots,c_r,1+a;\ x_1,\cdots,x_r,y\right] t^n$$

$$= (1-t)^{w-a-1}(1-t+yt)^{-w}\exp\left(\frac{-xt}{1-t}\right) F_A^{(r+1)}\left[w: b_1,\cdots,b_r,-;\right.$$

$$\left. c_1,\cdots,c_r,1+a;\ \frac{1-t}{1-t+yt}x_1,\cdots,\frac{1-t}{1-t+yt}x_r,\ \frac{xyt}{(1-t)(1-t+yt)}\right].$$

Proof of (6.2.1) : In view of the definition (1.4.5) of $\Psi_2^{(r)}$, we can write

$$\sum_{n=0}^{\infty} \frac{(w)_n}{(1+a)_n} L_n^{(a)}(x)\, \Psi_2^{(r)}\left[w+n : b_1,\cdots,b_r ;\ x_1,\cdots,x_r\right] t^n$$

$$= \sum_{n,m_1,\cdots,m_r=0}^{\infty} \frac{(w)_n}{(1+a)_n} L_n^{(a)}(x)\, \frac{(w+n)_{m_1+\cdots+m_r}}{(b_1)_{m_1}\cdots(b_r)_{m_r}} \cdot \frac{x_1^{m_1}\cdots x_r^{m_r}}{m_1!\cdots m_r!}\, t^n$$

$$= \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(w)_{m_1+\cdots+m_r}}{(b_1)_{m_1}\cdots(b_r)_{m_r}} \cdot \frac{x_1^{m_1}\cdots x_r^{m_r}}{m_1!\cdots m_r!} \sum_{n=0}^{\infty} \frac{(w+m_1+\cdots m_r)_n}{(1+a)_n} L_n^{(a)}(x) t^n .$$

Now applying the following well known result of Chaundy [35]

$$\sum_{n=0}^{\infty} \frac{(b)_n}{(1+a)_n} L_n^{(a)}(x) t^n = (1-t)^{-b}\, {}_1F_1(b;1+a;-xt/(1-t)) ,$$

we observe that

$$\sum_{n=0}^{\infty} \frac{(w)_n}{(1+a)_n} L_n^{(a)}(x)\, \Psi_2^{(r)}\left[w+n : b_1,\cdots,b_r ;\ x_1,\cdots,x_r\right]$$

$$= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(w)_{m_1+\ldots+m_r}}{(b_1)_{m_1}\ldots(b_r)_{m_r}} \cdot \frac{x_1^{m_1}\ldots x_r^{m_r}}{m_1!\ldots m_r!} (1-t)^{-w-m_1-\ldots-m_r}$$

$$\times \; {}_1F_1(w+m_1+\ldots+m_r \; ; \; 1+a \; ; \; -xt/(1-t))$$

$$= (1-t)^{-w} \sum_{m_1,\ldots,m_r,n=0}^{\infty} \frac{(w)_{m_1+\ldots+m_r+n}}{(b_1)_{m_1}\ldots(b_r)_{m_r}(1+a)_n \; m_1!\ldots m_r! \; n!}$$

$$\times \left(\frac{-x_1}{1-t}\right)^{m_1} \ldots \left(\frac{-x_r}{1-t}\right)^{m_r} \left(\frac{-xt}{1-t}\right)^{n}$$

$$= (1-t)^{-w} \; \Psi_2^{(r+1)}\left[w: b_1,\ldots,b_r, 1+a \; ; \; \frac{x_1}{1-t}, \ldots, \frac{x_r}{1-t}, \frac{-xt}{1-t}\right],$$

which completes the proof of (6.2.1).

Special Cases : (i) If we write $w=a$, $1+a = c$, $t=x$, $x=y$ and $x_1 = \ldots = x_r = 0$ in (6.2.1), we obtain

$$(6.2.3) \qquad \sum_{n=0}^{\infty} \frac{(a)_n \, L_n^{(c-1)}(y)}{(c)_n} x^n = (1-x)^{-a} \; {}_1F_1(a;c;xy/(x-1)) ,$$

which is due to Deshpande and Bhise [36].

(ii) The substitution $w = 1+b+m$, $b_1 = 1+b$, $r=1$ and $x_1 = -y$, in (6.2.1), we get

$$(6.2.4) \qquad \sum_{n=0}^{\infty} \frac{(1+b+m)_n}{(1+a)_n} L_n^{(a)}(x) \; {}_1F_1(1+b+m+n \; ; \; 1+b \; ; \; -y) \; t^n$$

$$= (1-t)^{-1-b-m} \Psi_2 [1+b+m : 1+b, 1+a ; \frac{y}{t-1}, \frac{xt}{t-1}],$$

which is probably a new result.

Further the use of Kummer's transformation [85, p.125]

(6.2.5) $${}_1F_1(a ; b ; z) = e^z {}_1F_1(b-a ; b ; -z),$$

on L.H.S. of (6.2.4) leads to another known result, due to Manocha [68, (14)].

(6.2.6) $$\sum_{n=0}^{\infty} \frac{(m+n)!}{(a+1)_n} L_{m+n}^{(b)}(y) L_n^{(a)}(x) t^n$$

$$= (1+b)_m (1-t)^{-1-b-m} e^y \Psi_2 [1+b+m : 1+b, 1+a ; \frac{y}{t-1}, \frac{xt}{t-1}]$$

This has also been derived by Srivastava and Singhal [112, (34)].

Proof of (6.2.2) : Replacing c by $w+m_1+ \ldots + m_r$ in the well known formula due to Brafman [19, p. 180, (15)] (also refer to Rainville [85, p.213] and Weisner's [139, p.1037, (4.6) with $\gamma = 1+a$])

$$\sum_{n=0}^{\infty} L_n^{(a)}(x) \, {}_2F_1(-n, c ; a+1 ; y) t^n = (1-t)^{c-a-1} (1-t+yt)^{-c}$$

$$.\exp [-xt/(1-t)] \; {}_1F_1(c ; a+1 ; xyt/(1-t)(1-t+yt))$$

we get

$$(6.2.8)\quad \sum_{n=0}^{\infty} L_n^{(a)}(x)t^n \sum_{m=0}^{n} \frac{(-n)_m\,(w+m_1+\cdots+m_r)_m}{(a+1)_m\, m!}\, y^m$$

$$= (1-t)^{w+m_1+\cdots+m_r-a-1}\,(1-t+yt)^{-w-m_1-\cdots-m_r}$$

$$\times \exp\left(\frac{-xt}{1-t}\right) \sum_{m=0}^{\infty} \frac{(w+m_1+\cdots+m_r)_m}{(a+1)_m\, m!} \left[\frac{xyt}{(1-t)(1-t+yt)}\right]^m .$$

Now

$$\sum_{n=0}^{\infty} L_n^{(a)}(x)\, F_A^{(r+1)}[w;b_1,\cdots,b_r,-n;c_1,\cdots,c_r,1+a;x_1,\cdots,x_r,y]\,t^n$$

$$= \sum_{n=0}^{\infty} L_n^{(a)}(x)t^n \sum_{m_1,\cdots,m_r=0}^{\infty} \sum_{m=0}^{n} (w)_{m_1+\cdots+m_r+m}$$

$$\times \frac{(b_1)_{m_1}\cdots(b_r)_{m_r}\,(-n)_m}{(c_1)_{m_1}\cdots(c_r)_{m_r}\,(1+a)_m} \cdot \frac{x_1^{m_1}\cdots x_r^{m_r}\,y^m}{m_1!\cdots m_r!\; m!}$$

$$= \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(w)_{m_1+\cdots+m_r}\,(b_1)_{m_1}\cdots(b_r)_{m_1}}{(c_1)_{m_1}\cdots(c_r)_{m_r}} \cdot \frac{x_1^{m_1}\cdots x_r^{m_r}}{m_1!\cdots m_r!}$$

$$\times \left[\sum_{n=0}^{\infty} L_n^{(a)}(x)t^n \sum_{m=0}^{n} \frac{(-n)_m\,(w+m_1+\cdots+m_r)_m}{(1+a)_m\, m!}\, y^m\right].$$

On summing up the inner series with the help of (6.2.8), the R.H.S. above becomes

$$= (1-t)^{w-a-1} (1-t+yt)^{w} \exp\left[-xt/(1-t)\right]$$

$$\times \sum_{m_1,\ldots,m_r,m=0}^{\infty} \frac{(w)_{m_1+\cdots+m_r} (b_1)_{m_1}\cdots(b_r)_{m_r} (w+m_1+\cdots+m_r)_m}{(c_1)_{m_1}\cdots(c_r)_{m_r} (1+a)_m\, m_1!\cdots m_r!\, m!}$$

$$[x_1(1-t)/(1-t+yt)]^{m_1}\cdots[x_r(1-t)/(1-t+yt)]^{m_r}[xyt/(1-t)(1-t+yt)]^{m},$$

which by the definition of $F_A^{(r)}$ is the R.H.S. of (6.2.2).

Thus is proved.

Special Cases : (i) Putting $x_1= \cdots =x_r=0$, and taking $\lim |w| \longrightarrow \infty$ in (6.2.2), we obtain the well known Hille-Hardy formula [85, p. 212]

$$(6.2.9) \quad \sum_{n=0}^{\infty} \frac{n!}{(a+1)_n} L_n^{(a)}(x)\, L_n^{(a)}(y)\, t^n$$

$$= (1-t)^{-a-1} \exp\left(\frac{-(x+y)t}{1-t}\right) {}_0F_1(-;a+1;xyt/(1-t)^2),$$

provided $|t|<1$ and 'a' is not a negative integer.

(ii) Next by the substitution $x=0$ and $a+1=u$ in (6.2.2), it follows that

$$(6.2.10) \quad \sum_{n=0}^{\infty} \frac{(u)_n}{n!} F_A^{(r+1)}[w:b_1,\cdots,b_r,-n;c_1,\cdots,c_r,u;x_1,\cdots,x_r,y]\,t^n$$

$$= (1-t)^{w-u} (1-t+yt)^{-w} \exp[-xt/(1-t)]$$

$$\times F_A^{(r)}\left[w; b_1,\cdots,b_r; c_1,\cdots,c_r; \frac{x_1(1-t)}{(1-t+yt)},\cdots,\frac{x_r(1-t)}{(1-t+yt)}\right].$$

6.3. Certain Bilateral Generating Relations Involving Jacobi Polynomials : Employing the same technique as used in the previous section, we establish following bilateral generating relations involving Jacobi polynomials and Lauricella Functions. We have also discussed some of their special cases :

(6.3.1)

$$\sum_{n=0}^{\infty} \frac{(w)_n}{(-a-b)_n} P_n^{(a-n,b-n)}(x)\, t^n$$

$$\times F_A^{(r)}\left[w+n: c_1,\cdots,c_r; d_1,\cdots,d_r; y_1,\cdots,y_r\right]$$

$$= U^{-w} F_A^{(r+1)}\left[w: c_1,\cdots,c_r,-b; d_1,\cdots,d_r,-a-b; \frac{y_1}{U},\cdots,\frac{y_r}{U},\frac{t}{U}\right],$$

(6.3.2)
$$\sum_{n=0}^{\infty} \frac{(w)_n}{(-a-b)_n} P_n^{(a-n,b-n)}(x)\, t^n$$

$$\times F_C^{(r)}\left[(w+n)/2 : (w+n+1)/2; c_1,\cdots,c_r; y_1^2,\cdots,y_r^2\right]$$

$$= V^{-w} F_A^{(r+1)}\left[w: c_1-1/2,\cdots,c_r-1/2,-b; 2c_1-1,\cdots,2c_r-1,\right.$$

$$\left. -a-b; 2y_1/v,\cdots, 2y_r/v, t/v\right]$$

and

$$(6.3.3)\quad \sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(a-n,\,b-n)}(x)\; t^n$$

$$\times\; F_A^{(r)}[-n: c_1,\cdots,c_r;\; d_1,\cdots,d_r;\; y_1,\cdots,y_r\,]$$

$$= \left(\frac{1+x}{2}\right)^m \left(\frac{x-1}{x+1}\right)^{-a} R^{a+b+m}\, \frac{(1+a+b+m)_m}{m!}\, F_A^{(r+1)}\Big[-a-b-m :$$

$$c_1,\cdots,c_r,-a-m;\, d_1,\cdots,d_r,-a-b-2m;\; \frac{t(x-1)}{2R}y_1,\cdots,\frac{t(x-1)}{2R}y_r,\frac{2}{R(1+x)}\,\Big]$$

where $U = 1 + (1+x)t/2$, $V=1+y_1+\cdots+y_r + (1+x)t/2$

and $R = 1 + (x-1)t/2$.

In our analysis we need the following results ([67], [43], [70]) :

$$(6.3.4)\quad \sum_{n=0}^{\infty} \frac{(w)_n}{(-a-b)_n} P_n^{(a-b,\,b-n)}(x)\; t^n$$

$$= \left(1+\tfrac{1}{2}(1+x)\,t\right)^{-w} {}_2F_1\left[\begin{matrix} w\;,\; -b; \\ -a-b\;; \end{matrix}\; \frac{t}{1+\frac{1}{2}(1+x)\,t}\right],$$

$$(6.3.5)\quad F_C^{(r)}[w/2:(1+w)/2\;;\; c_1,\cdots,c_r\;;\; x_1^2,\cdots,x_r^2\,]$$

$$= (1+x_1+\cdots+x_r)^{-w}\, F_A^{(r)}\Big[w: c_1-1/2,\cdots,c_r-1/2\;;$$

$$2c_1-1,\cdots,2c_r-1;\; \frac{2x_1}{1+x_1+\cdots+x_r},\cdots,\frac{2x_r}{1+x_1+\cdots+x_r}\Big]$$

and

$$(6.3.6)\quad \sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(a-n,b-n)}(x)\, t^n = R^{a+b+m} \left(\frac{x+1}{2}\right)^m \left(\frac{x-1}{x+1}\right)^{-a}$$

$$\times \frac{(1+a+b+m)_m}{m!}\, {}_2F_1\left[\begin{matrix} -a-b-m, -a-m ; \\ -a-b-2m ; \end{matrix}\ \frac{2}{R(x+1)}\right].$$

Proof of (6.3.1). From the definition (1.4.1), we have

$$\sum_{n=0}^{\infty} \frac{(w)_n}{(-a-b)_n} P_n^{(a-n,b-n)}(x)\, t^n\, F_A^{(r)}\left[w+n: c_1, \dots, c_r; d_1, \dots, d_r; y_1, \dots, y_r\right]$$

$$= \sum_{m_1,\dots,m_r=0}^{\infty} \frac{(w)_{m_1+\dots+m_r} (c_1)_{m_1} \dots (c_r)_{m_r}}{(d_1)_{m_1} \dots (d_r)_{m_r}} \cdot \frac{y_1^{m_1} \dots y_r^{m_r}}{m_1! \dots m_r!}$$

$$\times \sum_{n=0}^{\infty} \frac{(w+m_1+\dots+m_r)_n}{(-a-b)_n} P_n^{(a-n,b-n)}(x)\, t^n,$$

the R.H.S. on using (6.3.4) and assuming $1+(1+x)t/2 = U$, becomes

$$= \sum_{m_1,\dots,m_r=0}^{\infty} \frac{(w)_{m_1+\dots+m_r} (c_1)_{m_1} \dots (c_r)_{m_r}}{(d_1)_{m_1} \dots (d_r)_{m_r}} \cdot \frac{y_1^{m_1} \dots y_r^{m_r}}{m_1 \dots m_r!}$$

$$\times\ U^{-w-m_1-\dots-m_r}\ {}_2F_1(w+m_1+\dots+m_r, -b; -a-b; t/U)$$

$$= U^{-w} \sum_{m_1,\dots,m_r,m=0}^{\infty} \frac{(w)_{m_1+\dots+m_r} (c_1)_{m_1} \dots (c_r)_{m_r} (-b)_m}{(d_1)_{m_1} \dots (d_r)_{m_r} (-a-b)_m\, m_1! \dots m_r!\, m!}$$

$$\times\ (y_1/U)^{m_1} \dots (y_r/U)^{m_r} (t/U)^m$$

$$= U^{-w}\; F_{\Lambda}^{(r+1)}\left[w: c_1,\cdots,c_r,-b; d_1,\cdots,d_r,-a-b;\ \frac{y_1}{U},\cdots,\ \frac{y_r}{U}\ ,\ \frac{t}{U}\right]$$

which completes the proof of (6.3.1).

Special Cases : (i) If we replace a,b,w,x and t respectively by b-c, -b, a, y and -x, and taking $y_1 = \cdots = y_r = 0$ in the above result (6.3.1), we obtain

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\ (-x)^n\ P_n^{(b-c-n,-b-n)}(y) = (1-\tfrac{1}{2}(1+y)x)^{-a}\ {}_2F_1\left[\begin{matrix} a, b; & -x \\ c; & \overline{1-\frac{1}{2}(1+y)} \end{matrix}\right].$$

The use of well known transformation

$$(6.3.7)\qquad F_2[a: b, b'; b, c; x, y] = (1-x)^{-a}\ {}_2F_1(a, b'; c; y/(1-x)),$$

leads to the following result of Despande and Bhise [36, (3.1)] :

$$(6.3.8)\qquad \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\ (-x)^n\ P_n^{(b-c-n,-b-n)}(x)$$

$$= (1-x)^{-a}\ F_2[1: b, b'; c, b'\ ;\ \frac{-x}{1-x}\ ,\ -\ \frac{x(1-y)}{2(1-x)}\ ]$$

(ii) In case we write $c_1 = c$, $c_2 = c'$, $d_1 = d$, $d_2 = d'$, $y_1 = y$, $y_2 = z$, $y_3 = \cdots = y_r = 0$ and replacing x,t,a and b, respectively by -x,-t, b and a, in (6.3.1), we get following result due to Manocha [67, (2.2)]

$$(6.3.9)\quad \sum_{n=0}^{\infty} \frac{(w)_n}{(-a-b)_n} t^n P_n^{(a-n,b-n)}(x)\; F_2[w+n:c,c';d,d';y,z]$$

$$= (1-t(1-t)/2)^{-w} F_A^{(3)}[w:c,c',-a;d,d',-a-b;$$

$$\frac{y}{1-\frac{1}{2}t(1-x)}, \frac{z}{1-\frac{1}{2}t(1-x)}, \frac{-t}{1-\frac{1}{2}t(1-x)}].$$

(iii) Next, if we let $x=1$, $-a=u$, $-a-b=v$ and $t=-t$, we obtain the generating relation

$$(6.3.10)\quad \sum_{n=0}^{\infty} \frac{(w)_n (u)_n}{(v)_n\, n!} t^n F_A^{(r)}[w+n:c_1,\cdots,c_r;d_1,\cdots,d_r;y_1,\cdots,y_r]$$

$$= (1-t)^{-w} F_A^{(r+1)}[w:c_1,\cdots,c_r, v-u;d_1,\cdots,d_r,v;$$

$$y_1/(1-t),\cdots,y_r/(1-t),-t/(1-t)].$$

<u>Proof of (6.3.2) and (6.3.3)</u>. Starting from the L.H.S. of (6.3.2) and making the use of (6.3.5), we have

$$\sum_{n=0}^{\infty} \frac{(w)_n}{(-a-b)_n} P_n^{(a-n,b-n)}(x) t^n$$

$$\times\; F_C^{(r)}[(w+n)/2:(w+n+1)/2;\; c_1,\cdots,c_r;\; y_1^2,\ldots,y_r^2]$$

$$= (1+y_1+\cdots+y_r)^{-w} \sum_{n=0}^{\infty} \frac{(w)_n}{(-a-b)_n} P_n^{(a-n,b-n)}(x) \left(\frac{t}{1+y_1+\cdots+y_r}\right)^n$$

$$\times F_A^{(r)}\left[w:c_1-\tfrac{1}{2},\cdots,c_r-\tfrac{1}{2};\ 2c_1-1,\cdots,2c_r-1;\right.$$

$$\left.2y_1/(1+y_1+\cdots+y_r),\cdots,2y_r/(1+y_1+\cdots+y_r)\right].$$

In view of the result (6.3.1) the R.H.S. becomes

$$v^{-w}\ F_A^{(r+1)}\left[w:c_1-\tfrac{1}{2},\cdots,c_r-\tfrac{1}{2},-b;\ 2c_1-1,\cdots,2c_r-1,-a-b;\right.$$

$$\left.2y_1/v,\cdots,2y_r/v,t/v\right].$$

Hence the result (6.3.2) is proved.

Further using the well known generating relation (6.3.6) and taking $1+(x-1)t/2=R$, the L.H.S. of (6.3.3) becomes

$$\sum_{m_1,\cdots,m_r=0}^{\infty}\frac{(c_1)_{m_1}\cdots(c_r)_{m_r}}{(d_1)_{m_1}\cdots(d_r)_{m_r}}\cdot\frac{(m+m_1+\cdots+m_r)!}{m!\,m_1!\cdots m_r!}\ y_1^{m_1}\cdots y_r^{m_r}$$

$$\times\ R^{a+b+m-m_1-\cdots-m_r}\left[\frac{x+1}{2}\right]^{m+m_1+\cdots+m_r}\left[\frac{x-1}{x+1}\right]^{-a+m_1+\cdots+m_r}$$

$$\times\ \frac{(1+a+b+m-m_1-\cdots-m_r)_{m+m_1+\cdots+m_r}}{(m+m_1+\cdots+m_r)!}$$

$$\times\ {}_2F_1\left[\begin{matrix}-a-b-m+m_1+\cdots+m_r,-a-b; & \dfrac{2}{R(x+1)}\\ -a-b-2m; & \end{matrix}\right]$$

$$=\left(\frac{x+1}{2}\right)^m\left(\frac{x-1}{x+1}\right)^{-a}R^{a+b+m}\ \frac{(1+a+b+m)_m}{m!}$$

$$\times \sum_{m_1,\cdots,m_r,p=0}^{\infty} \frac{(-a-b-m)_{m_1+\cdots+m_r} (c_1)_{m_1}\cdots(c_r)_{m_r} (-a-b-m+m_1+\cdots+m_r)_p}{(d_1)_{m_1}\cdots(d_r)_{m_r} (-a-b-2m)_p \, m_1!\cdots m_r! \, p!}$$

$$\times \left(\frac{x-1}{2R} ty_1\right)^{m_1} \cdots \left(\frac{x-1}{2R} ty_r\right)^{m_r} \left(\frac{2}{R(x+1)}\right)^p$$

$$= \left(\frac{x+1}{2}\right)^m \left(\frac{x-1}{x+1}\right)^{-a} R^{a+b+m} \frac{(1+a+b+m)_m}{m!} F_A^{(r+1)} \Big[-a-b-m :$$

$$c_1,\cdots,c_r,-a-m; d_1,\cdots,d_r,-a-b-2m; \frac{t(x-1)}{2R} y_1,\cdots, \frac{t(x-1)}{2R} y_r, \frac{2}{R(1+x)} \Big].$$

Thus (6.3.3) is proved.

Special Cases : (i) Putting $c_1=c$, $c_2=d$, $d_1=e$, $d_2=f$, $y_1=y$, $y_2=z$ and $y_3=\cdots=y_r=0$ in (6.3.3), we get the following bilateral generating function proved earlier by Manocha [69, (2.2)]

$$(6.3.11) \quad \sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(a-n,b-n)}(x) \, F_2[-n:c,d;e,f;y,z] \, t^n$$

$$= \frac{(1+a+b+m)_m}{m!} \left(\frac{x+1}{2}\right)^m \left(\frac{x-1}{x+1}\right)^{-a} R^{a+b+m}$$

$$\times F_A^{(3)} \Big[-a-b-m:-a-m,c,d;-a-b-2m,e,f;$$

$$\frac{2}{R(x+1)}, -\frac{yt(1-x)}{2R}, -\frac{zt(1-x)}{2R} \Big].$$

(ii) If we write $y_1 = 2/(1-y)$, $c_1=-c$, $d_1=-c-d$ and $y_2=\cdots=y_r=0$, we have

$$\sum_{n=0}^{\infty} \frac{(m+n)!}{n!} P_{m+n}^{(a-n,b-n)}(x) \; {}_2F_1\left[\begin{matrix} -n,-c \; ; & \frac{2}{1-y} \\ -c-d \; ; & \end{matrix}\right]$$

$$= (1+a+b+m)_m \left(\frac{1+x}{2}\right)^m \left(\frac{x-1}{x+1}\right)^{-a} \left[1+(x-1)t/2\right]^{a+b+m}$$

$$\times \; F_2 \left[-a-b-m: -c, -a-m; -c-d, -a-b-2m \; ;\right.$$

$$\left.\frac{t(x-1}{(1-y)(1+\frac{1}{2}(x-1)t)}, \frac{2}{(1+x)(1+\frac{1}{2}(x-1)t)}\right],$$

now replacing t by $t(1-y)/2$, we get

$$(6.3.12) \quad \sum_{n=0}^{\infty} \frac{(m+n)!}{(-c-d)_n} P_{m+n}^{(a-n,b-n)}(x) \; P_n^{(c-n,d-n)}(y) \, t^n$$

$$= (1+a+b+m)_m \left(\frac{x+1}{2}\right)^m \left(\frac{x+1}{x-1}\right)^a \left[1-(x-1)(y-1)t/4\right]^{a+b+m}$$

$$\times \; F_2 \left[-a-b-m: -c, -a-m \; ; -c-d, -a-b-2m \; ;\right.$$

$$\left.\frac{(x-1)t}{2(1-\frac{1}{4}(x-1)(y-1)t)}, \frac{2}{(x+1)(1-\frac{1}{4}(x-1)(y-1)t)}\right].$$

Further the substitution $m=0$ and the use of the transformation of Appell and Kampe de Fériet [15, p.35, (10)]

$$F_2[a: b, b'; a, c'; x, y] = (1-x)^{-b} F_1[b' : b, a-b; c' \; ; \frac{y}{1-x}, y] \text{ in } (6.3.12),$$

yields the following result established earlier by Manocha and Sharma [70, (13)]

$$(6.3.13)\quad \sum_{n=0}^{\infty} \frac{n!}{(-c-d)_n} P_n^{(a-n,b-n)}(x)\, P_n^{(c-n,d-n)}(y)\, t^n$$

$$= [1-(x+1)(y+1)t/4]^a [1-(x-1)(y+1)t/4]^b$$

$$\times\ F_1\left[-d:-a,-b;-c-d\,;\ \frac{2(x+1)t}{(x+1)(y+1)t-4},\ \frac{2(x-1)t}{(x-1)(y+1)t-4}\right].$$

6.4 <u>Certain Multiple Generating Functions Involving Laguerre Polynomials</u> : By making use of the results of section 6.2, we establish two multiple generating relations for Laguerre polynomials. The results thus obtained also involve Lauricella functions and not only provides the extension of the bilateral generating functions (6.2.1) and (6.2.2) but are also very general in nature :

$$(6.4.1)\quad \sum_{n_1,\cdots,n_s=0}^{\infty} \frac{(w)_{n_1+\cdots+n_s}}{(1+a_1)_{n_1}\cdots(1+a_s)_{n_s}} L_{n_1}^{(a_1)}(x_1)\cdots L_{n_s}^{(a_s)}(x_s)$$

$$\times\ \Psi_2^{(r)}[w+n_1+\cdots+n_s : b_1,\cdots,b_r;\ y_1,\cdots,y_r]\, t_1^{n_1}\cdots t_s^{n_s}$$

$$= D_s^{-w}\ \Psi_2^{(r+s)}[w: b_1,\cdots,b_r, 1+a_1,\cdots,1+a_s\,;$$

$$y_1/D_s,\cdots,y_r/D_s, -x_1t_1/D_s,\cdots,-x_st_s/D_s\,] ,$$

where $D_s = 1 - \sum_{j=1}^{s} t_j$; $s = 1,2,3,\ldots,$

and

$$(6.4.2) \quad \sum_{n_1,\ldots,n_s=0}^{\infty} L_{n_1}^{(a_1)}(x_1)\ldots L_{n_s}^{(a_s)}(x_s)\, t_1^{n_1}\ldots t_s^{n_s}$$

$$\times\ F_A^{(s)}[w: -n_1,\ldots,-n_s;\ 1+a_1,\ldots+1+a_s;\ y_1,\ldots,y_s]$$

$$= T^{w}(1-t_1)^{-1-a_1}\ldots(1-t_s)^{-1-a_s}\exp[-\frac{x_1t_1}{1-t_1}-\ldots-\frac{x_st_s}{1-t_s}]$$

$$\times\ \Psi_2^{(s)}[w: 1+a_1,\ldots,1+a_s;\ \frac{x_1y_1t_1T}{(1-t_1)^2},\ldots,\frac{x_sy_st_sT}{(1-t_s)^2}].$$

where $T = \prod_{j=1}^{s}\left(\frac{1-t_j}{1-t_j+y_jt_j}\right)$; $s = 1,2,3,\ldots,$.

<u>Proof of (6.4.1)</u> : We prove this result by method of induction. From (6.2.1) we see that result is true for s=1. Let us assume that for $s=k$ (6.4.1) holds, that is

$$\sum_{n_1,\ldots,n_k=0}^{\infty} \frac{(w)_{n_1+\cdots+n_k}}{(1+a_1)_{n_1}\ldots(1+a_k)_{n_k}} L_{n_1}^{(a_1)}(x_1)\ldots L_{n_k}^{(a_k)}(x_k)$$

$$\times\ \Psi_2^{(r)}[w+n_1+\cdots+n_k : b_1,\ldots,b_r;\ y_1,\ldots,y_r]\,t_1^{n_1}\ldots t_k^{n_k}$$

$$= D_k^{-w}\,\Psi_2^{(r+k)}[w: b_1,\ldots,b_r, 1+a_1,\ldots,1+a_k; y_1/D_k\ldots y_r/D_k,$$

$$-x_1t_1/D_k,\ldots,-x_kt_k/D_k].$$

Now replacing w by $w+n_{k+1}$, multiplying by

$$(w)_{n_{k+1}}\, t^{n_{k+1}}\, L_{n_{k+1}}^{(a_{k+1})}(x_{k+1}) \,/\, (1+a_{k+1})_{n_{k+1}}$$ and

summing the series (after adjusting parameters) from $n_{k+1}=0$ to ∞, we get

$$\sum_{n_1,\cdots,n_k,n_{k+1}=0}^{\infty} \frac{(w)_{n_1+\cdots+n_k+n_{k+1}}}{(1+a_1)_{n_1}\cdots(1+a_k)_{n_k}(1+a_{k+1})_{n_{k+1}}} L_{n_1}^{(a_1)}(x_1)\cdots$$

$$L_{n_k}^{(a_k)}(x_k)\, L_{n_{k+1}}^{(a_{k+1})}(x_{k+1})\, t_1^{n_1}\cdots t_k^{n_k}\, t_{k+1}^{n_{k+1}}$$

$$\times\ \Psi_2^{(r)}[w+n_1+\cdots+n_k+n_{k+1} : b_1,\cdots,b_r ;\ y_1,\cdots,y_r]$$

$$= D_k^{-w} \sum_{n_{k+1}=0}^{\infty} \frac{(w)_{n_{k+1}}}{(1+a_{k+1})_{n_{k+1}}} \left(\frac{t_{k+1}}{D_k}\right)^{n_{k+1}} L_{n_{k+1}}^{(a_{k+1})}(x_{k+1})$$

$$\times\ \Psi_2^{(r+k)}[w+n_{k+1} : b_1,\cdots,b_r, 1+a_1,\cdots,1+a_k;\ y_1/D_k,\cdots,$$

$$y_r/D_k, -x_1t_1/D_k,\cdots,-x_kt_k/D_k] ,$$

which after using (6.2.1), gives us

$$\text{L.H.S.} = D_{k+1}^{-w}\, \Psi_2^{(r+k+1)}[w: b_1,\cdots,b_r, 1+a_1,\cdots,1+a_{k+1} ;$$

$$y_1/D_{k+1},\cdots,y_r/D_{k+1}, -x_1t_1/D_{k+1},\cdots,-x_{k+1}\ t_{k+1}/D_{k+1}].$$

Thus, (6.4.1) holds by mathematical induction.

Remark : If we let $r=1$, $y_1=-y$, $w=b+m+1$, and $b_1=b+1$ in (6.4.1), we get

$$\sum_{n_1,\ldots,n_s=0}^{\infty} \frac{(1+b+m)_{n_1+\cdots+n_s}}{(1+a_1)_{n_1}\cdots(1+a_s)_{n_s}} L_{n_1}^{(a_1)}(x_1)\ldots L_{n_s}^{(a_s)}(x_s)$$

$$\times\ t_1^{n_1}\ldots t_s^{n_s}\ {}_1F_1(1+b+m+n_1+\cdots+n_s\ ;\ 1+b; -y)$$

$$= D_s^{-b-m-1}\ \Psi_2^{(s+1)}\left[b+m+1: b+1, a_1+1,\ldots,a_s+1;\ \frac{-y}{D_s},\ \frac{-t_1x_1}{D_s},\ldots,\frac{-t_sx_s}{D_s}\right],$$

which after using Kummar's formula (6.2.5), we get the following multilinear generating function due to Srivastava and Singhal [124, (5)].

$$(6.4.4)\quad \sum_{n_1,\ldots,n_s=0}^{\infty} \frac{(m+n_1+\cdots+n_s)!}{(a_1+1)_{n_1}\cdots(a_s+1)_{n_s}} L_{m+n_1+\cdots+n_s}^{(b)}(y)$$

$$\times\ L_{n_1}^{(a_1)}(x_1)\ldots L_{n_s}^{(a_s)}(x_s)\, t_1^{n_1}\ldots t_s^{n_s}$$

$$= (a+1)_m\ D_s^{-b-m-1}\ \Psi_2^{(s+1)}\left[b+m+1: b+1, a_1+1,\ldots,a_s+1\ ;\right.$$

$$\left.-y/D_s, -t_1x_1/D_s,\ldots,-t_sx_s/D_s\ \right].$$

Proof of (6.4.2) : Starting from L.H.S. of (6.4.2) and taking the help of (6.2.2), we have

$$\sum_{n_1,\ldots,n_s=0}^{\infty} t_1^{n_1}\ldots t_s^{n_s} L_{n_1}^{(a_1)}(x_1)\ldots L_{n_s}^{(a_s)}(x_s)$$

$$X \quad F_A^{(s)}\left[w:-n_1,\ldots,-n_s\ ;\ 1+a_1,\ldots,1+a_s\ ;\ y_1,\ldots,y_s\right]$$

$$= (1-t_1)^{w-a_1-1}(1-t_1+y_1t_1)^{-w} \exp\left[-xt_1/(1-t_1)\right]$$

$$X \sum_{n_2,\ldots,n_s=0}^{\infty} t_2^{n_2}\ldots t_s^{n_s} L_{n_2}^{(a_2)}(x_2)\ldots L_{n_s}^{(a_s)}(x_s)$$

$$X \quad F_A^{(s)}\left[w:-,-n_2,\ldots,-n_s : 1+a_1,\ldots,1+a_s;\right.$$

$$\left. x_1y_1t_1T_1/(1-t_1)^2,\ y_2T_1,\ldots,y_sT_1\right]\ ;$$

where $T_1 = (1-t_1)/(1-t_1+y_1t_1)$.

The (s-1) times repeatition of this process would give required result (6.4.2).

Remark : In particular if we take $1+a_1=a_1,\ldots,1+a_s=a_s$, $x_1=\ldots=x_s=0$ and use the well known result $L_n^{(a)}(0) = (1+a)_n/n!$ in (6.4.2), we obtain an interesting multiple generating relation

(6.4.5) $$\sum_{n_1,\ldots,n_s=0}^{\infty} \frac{(a_1)_{n_1}\ldots(a_s)_{n_s}}{n_1!\ldots n_s!} t_1^{n_1}\ldots t_s^{n_s}$$

$$X \quad F_A^{(s)}\left[w:-n_1,\ldots,-n_s\ ;\ a_1,\ldots,a_s\ ;\ y_1,\ldots,y_s\right]$$

$$= (1-t_1)^{-a_1} \dots (1-t_s)^{-a_s} \left[\left(\frac{1-t_1}{1-t_1+y_1 t_1} \right) \dots \left(\frac{1-t_s}{1-t_s+y_s t_s} \right) \right].$$

6.5 Multiple Generating Relations Involving Jacobi Polynomials:

Under this section we derive following multiple generating functions involving Jacobi polynomials instead of Laguerre polynomials :

$$(6.5.1) \quad \sum_{n_1,\dots,n_s=0}^{\infty} \frac{(w)_{n_1+\dots+n_s} \, (u)_{n_1+\dots+n_s}}{(a_1+1)_{n_1} \dots (a_s+1)_{n_s} \, (b_1+1)_{n_1} \dots (b_s+1)_{n_s}}$$

$$\times \; t_1^{n_1} \dots t_s^{n_s} \; P_{n_1}^{(a_1,b_1)}(x_1) \dots P_{n_s}^{(a_s,b_s)}(x_s)$$

$$\times \; F_C^{(r)} [w+n_1+\dots+n_s : u+n_1+\dots+n_s ; c_1,\dots,c_r ; y_1,\dots,y_r]$$

$$= F_C^{(r+2s)} [w : u ; c_1,\dots,c_r, a_1+1,\dots,a_s+1, b_1+1,\dots,b_s+1 ;$$

$$y_1,\dots,y_r, \tfrac{1}{2}(x_1-1)t_1,\dots,\tfrac{1}{2}(x_s-1)t_s, \tfrac{1}{2}(x_1+1)t_1,\dots,\tfrac{1}{2}(x_s+1)t_s].$$

$$(6.5.2) \quad \sum_{n_1,\dots,n_s=0}^{\infty} \frac{(w)_{n_1+\dots+n_s}}{(u)_{n_1+\dots+n_s}} \; P_{n_1}^{(a_1-n_1,b_1-n_1)}(x_1) \dots P_{n_s}^{(a_s-n_s,b_s-n_s)}(x_s)$$

$$\times \; t_1^{n_1} \dots t_s^{n_s} \; F_D^{(r)} [w+n_1+\dots+n_s : c_1,\dots,c_r ; u+n_1+\dots+n_s ; y_1,\dots,y_r]$$

$$= F_D^{(r+2s)} [w : c_1,\dots,c_r, -a_1,\dots,-a_s, -b_1,\dots,-b_s ; u ; y_1,\dots,y_r ,$$

$$-\tfrac{1}{2}(x_1+1)t_1,\dots,-\tfrac{1}{2}(x_s+1)t_s, -\tfrac{1}{2}(x_1-1)t_1,\dots,\tfrac{1}{2}(x_s-1)t_s]$$

and

$$(6.5.3)\quad \sum_{n_1,\cdots,n_s=0}^{\infty} \frac{(w)_{n_1+\cdots+n_s}}{(-a_1-b_1)_{n_1}\cdots(-a_s-b_s)_{n_s}} t_1^{n_1}\cdots t_s^{n_s}$$

$$\times\ P_{n_1}^{(a_1-n_1,b_1-n_1)}(x_1)\cdots P_{n_s}^{(a_s-n_s,b_s-n_s)}(x_s)$$

$$\times\ F_A^{(r)}[w+n_1+\cdots+n_s : c_1,\cdots,c_r;\ d_1,\cdots,d_r;\ y_1,\cdots,y_r]$$

$$= (1+Z_s)^{-w}\ F_A^{(r+s)}[w:c_1,\cdots,c_r,-b_1,\cdots,-b_s;\ d_1,\cdots,d_r,$$

$$-a_1-b_1,\cdots,-a_s-b_s;\frac{y_1}{1+Z_s},\cdots,\frac{y_r}{1+Z_s},\frac{t_1}{1+Z_s},\cdots,\frac{t_s}{1+Z_s}],$$

where $Z_s = 1+\frac{1}{2}\sum_{j=1}^{s}(1+x_j)t_j\ ;\ s=1,2,3,\cdots,$.

<u>Proof of (6.5.1)</u>. Consider

$$\sum_{n_1,\cdots,n_s=0}^{\infty} \frac{(w)_{n_1+\cdots+n_s}(u)_{n_1+\cdots+n_s}}{(a_1+1)_{n_1}\cdots(a_s+1)_{n_s}\ (b_1+1)_{n_1}\cdots(b_s+1)_{n_s}}$$

$$\times\ t_1^{n_1}\cdots t_s^{n_s}\ P_{n_1}^{(a_1,b_1)}(x_1)\cdots P_{n_s}^{(a_s,b_s)}(x_s)$$

$$\times\ F_C^{(r)}[w+n_1+\cdots+n_s:u+n_1+\cdots+n_s;c_1,\cdots,c_r;y_1,\cdots,y_r]$$

$$\sum_{n_1,\cdots,n_s,m_1,\cdots,m_r=0}^{\infty} \frac{(w)_{n_1+\cdots+n_s+m_1+\cdots+m_r}(u)_{n_1+\cdots+n_s+m_1+\cdots+m_r}}{(a_1+1)_{n_1}\cdots(a_s+1)_{n_s}\ (b_1+1)_{n_1}\cdots(b_s+1)_{n_s}}$$

$$\times \frac{y_1^{m_1}\cdots y_r^{m_r}\; t_1^{n_1}\cdots t_s^{n_s}}{(c_1)_{m_1}\cdots(c_r)_{m_r}\; m_1!\cdots m_r!}\, P_{n_1}^{(a_1,b_1)}(x_1)\cdots P_{n_s}^{(a_s,b_s)}(x_s)$$

$$= \sum_{n_2,\ldots,n_s,m_1,\ldots,m_r=0}^{\infty} \frac{(w)_{n_2+\cdots+n_s+m_1+\cdots+m_r}\,(u)_{n_2+\cdots+n_s+m_1+\cdots+m_r}}{(a_2+1)_{n_2}\cdots(a_s+1)_{n_s}\,(b_2+1)_{n_2}\cdots(b_s+1)_{n_s}}$$

$$\times \frac{y_1^{m_1}\cdots y_r^{m_r}\; t_2^{n_2}\cdots t_s^{n_s}}{(c_1)_{m_1}\cdots(c_r)_{m_r}\; m_1!\cdots m_r!}\; P_{n_2}^{(a_2,b_2)}(x_2)\cdots P_{n_s}^{(a_s,b_s)}(x_s)$$

$$\times \sum_{n_1=0}^{\infty} \frac{(w+n_2+\cdots+n_s+m_1+\cdots+m_r)_{n_1}\,(u+n_2+\cdots+n_s+m_1+\cdots+m_r)_{n_1}}{(a_1+1)_{n_1}\,(b_1+1)_{n_1}}$$

$$\times\; t_1^{n_1}\, P_{n_1}^{(a_1,b_1)}(x_1)\,.$$

Using the Brafman's generating function [85, p. 271]

$$\sum_{n=0}^{\infty} \frac{(w)_n\,(u)_n}{(a+1)_n\,(b+1)_n} P_n^{(a,b)}(x)\, t^n = F_4\left[w; u; a+1, b+1, \tfrac{1}{2}(x-1)t, \tfrac{1}{2}(x+1)t\right]$$

$$= \sum_{p,q=0}^{\infty} \frac{(w)_{p+q}\,(u)_{p+q}}{(a+1)_p\,(b+1)_q\; p!\; q!}\, [(x-1)t/2]^p\, [(x+1)t/2]^q,$$

the R.H.S. becomes

$$\sum_{n_2,\ldots,n_s,m_1,\ldots,m_r,p,q=0}^{\infty} \frac{(w)_{n_2+\cdots+n_s+m_1+\cdots+m_r+p+q}\,(u)_{n_2+\cdots+n_s+m_1+\cdots+m_r+p+q}}{(a_2+1)_{n_2}\cdots(a_s+1)_{n_s}\,(b_2+1)_{n_2}\cdots(b_s+1)_{n_s}}$$

$$\times \frac{y_1^{m_1} \cdots y_r^{m_r} \; ((x_1-1)t_1/2)^p \; ((x_1+1)t_1/2)^q}{(c_1)_{m_1} \cdots (c_r)_{m_r} \; (a_1+1)_p \; (b_1+1)_q \; m_1! \cdots m_r! \; p! \; q!}$$

$$\times \; t_2^{n_2} \cdots t_s^{n_s} \; P_{n_2}^{(a_2,b_2)}(x_2) \cdots P_{n_s}^{(a_s,b_s)}(x_s).$$

Repeating the above process (s-1) times, we get (6.5.1).

<u>Proof of (6.5.2)</u> : The proof is similar as that of (6.5.1), but this time we make use of the generating function[11]

$$\sum_{n=0}^{\infty} \frac{(w)_n}{(u)_n} P_n^{(a-n,b-n)}(x)\, t^n = F_1[w: -a, -b; u; -(x+1)t/2, -(x-1)t/2]$$

In fact (6.5.1) and (6.5.2) are the extensions of the following results established by Saxena [87] and the same were later on obtained by Srivastava and Daust [118]. In particular, when $s=1$, $x_1=x$, $n_1=n$, $t_1=t$, $a_1=a$ and $b_1=b$ in (6.5.1) and (6.5.2), we get

$$(6.5.4) \quad \sum_{n=0}^{\infty} \frac{(w)_n \; (u)_n}{(a+1)_n \; (b+1)_n} P_n^{(a,b)}(x)\, t^n$$

$$\times \; F_C^{(r)}[w+n : u+n; c_1, \ldots, c_r ; y_1, \ldots, y_r]$$

$$= F_C^{(r+2)}[w: u; c_1, \ldots, c_r, a+1, b+1; y_1, \ldots, y_r, (x-1)t/2, (x+1)t/2]$$

and

$$(6.5.5) \quad \sum_{n=0}^{\infty} \frac{(w)_n}{(u)_n} P_n^{(a-n,b-n)} t^n$$

$$\times \quad F_D^{(r)}\left[w+n : c_1,\cdots,c_r;\ u+n;\ y_1,\cdots,y_r\right]$$

$$= F_D^{(r+2)}\left[w:c_1,\cdots,c_r,-a,-b;u;y_1,\cdots,y_r,-(x+1)t/2,-(x-1)t/2\right]$$

Next, if we take $r=1$, $c_1=1+a$, $y_1=(y-1)/(y+1)$, $w=1+a+m$, $u=1+a+b+m$ in (6.5.1), we get

$$(6.5.6)\quad \sum_{n_1,\cdots,n_s=0}^{\infty} \frac{(1+a+m)_{n_1+\cdots+n_s}\,(1+a+b+m)_{n_1+\cdots+n_s}}{(a_1+1)_{n_1}\cdots(a_s+1)_{n_s}\,(b_1+1)_{n_1}\cdots(b_s+1)_{n_s}}$$

$$\times\quad t_1^{n_1}\cdots t_s^{n_s}\ P_{n_1}^{(a_1,b_1)}(x_1)\cdots P_{n_s}^{(a_s,b_s)}(x_s)$$

$$\times\ {}_2F_1\left(1+a+m+n_1+\cdots+n_s,1+a+b+m+n_1+\cdots+n_s;1+a;\ \frac{y-1}{y+1}\right)$$

$$= F_C^{(2s+1)}\left[a+b+m+1,a+m+1;a+1,a_1+1,\cdots,a_s+1;b_1+1,\cdots,b_s+1\ ;\right.$$

$$\left.\frac{y-1}{y+1},\tfrac{1}{2}(x_1-1)t_1,\cdots,\tfrac{1}{2}(x_s-1),\tfrac{1}{2}(x_1+1)t_1,\cdots,\tfrac{1}{2}(x_s-1)t_s\right].$$

On employing the Euler's transformation [85, p.60]

$$(6.5.7)\quad {}_2F_1(a,b;c;z) = (1-z)^{-a}\ {}_2F_1(a,c-b;c;-z/(1-z)),$$

and taking $t_1=2u_1/(y+1),\cdots,t_s=2u_s/(y+1)$, after adjusting the parameters, we obtain the following multilinear generating relations due to Mandekar and Thakare [65,(3.1) p.713]

$$(6.5.8)\quad \sum_{n_1,\ldots,n_s=0}^{\infty} \frac{(a+b+m+1)_{n_1+\cdots+n_s}\,(m+n_1+\cdots+n_s)!}{(a_1+1)_{n_1}\,(b_1+1)_{n_1}\cdots(a_s+1)_{n_s}\,(b_s+1)_{n_s}}\, u_1^{n_1}\cdots u_s^{n_s}$$

$$\times\ P_{n_1}^{(a_1,b_1)}(x_1)\cdots P_{n_s}^{(a_s,b_s)}(x_s)\, P_{m+n_1+\cdots+n_s}^{(a,b)}(y)$$

$$= (a+1)_m\left(\frac{x+1}{2}\right)^{-a-b-m-1} F_C^{(2s+1)}\Big[a+b+m+1{:}\,a+m+1;\,a+1,a_1+1,\ldots,a_s+1,$$

$$b_1+1,\ldots,b_s+1;\,\frac{y-1}{y+1},\frac{(x_1-1)u_1}{y+1},\ldots,\frac{(x_s-1)u_s}{y+1},\frac{(x_1+1)u_1}{y+1},\ldots,\frac{(x_s+1)u_s}{y+1}\Big].$$

Further, if we write $r=1$, $w=1+a+m$, $u=1+a$, $c_1=1+a+b+m$, $y_1=(y-1)/(y+1)$ and $t_1=2u_1/(y+1),\ldots,t_s=2u_s/(y+1)$ in (6.5.2), we get an interesting result

$$(6.5.9)\quad \sum_{n_1,\ldots,n_s=0}^{\infty} P_{n_1}^{(a_1-n_1,\,b_1-n_1)}(x_1)\cdots P_{n_s}^{(a_s-n_s,\,b_s-n_s)}(x_s)$$

$$\times\ P_m^{(a+n_1+\cdots+n_s,\,b-n_1-\cdots-n_s)}(y)\, u_1^{n_1}\cdots u_s^{n_s}$$

$$= \frac{(1+a)_m}{m!}\left(\frac{y+1}{2}\right)^{-1-a-m} F_D^{(2s+1)}\Big[1+a+m{:}\,1+a+b+m,-a_1,\ldots,-a_s,$$

$$-b_1,\ldots,-b_s;\,1+a;\ \frac{y-1}{y+1},\ \frac{-(x_1+1)u_1}{y+1},\ldots,\ \frac{-(x_s+1)u_s}{y+1},$$

$$\frac{-(x_1-1)u_1}{y+1},\ldots,\ \frac{-(x_s-1)u_s}{y+1}\Big],$$

which is believed to be new.

Remark : In particular if we let $s=1$, $n_1=n$, $a_1=c$, $b_1=d$ and $x_1=x$ in (6.5.9), we get

$$(6.5.10)\quad \sum_{n=0}^{\infty} P_n^{(c-n,d-n)}(x)\, P_m^{(a+n,b-n)}(y) = \frac{(1+a)_m}{m!}\left(\frac{y+1}{2}\right)^{-1-a-m}$$

$$\times F_D^{(3)}\left[1+a+m: 1+a+b+m, -c, -d; 1+a;\ \frac{y-1}{y+1}, -\frac{(x+1)u}{y+1}, -\frac{(x-1)u}{y+1}\right].$$

Proof of (6.5.3) : The result can be proved easily by mathematical induction and details of the proof have been omitted.

Special cases of (6.5.3) : If we take $r=1$, $y_1=2/(x+1)$, $c_1=-a$, $d_1=-a-b$, replacing $a_1,\cdots,a_s, b_1,\cdots,b_s$, $x_1,\cdots,x_s$, $t_1,\cdots,t_s$ and w respectively by $d_1,\cdots,d_s$, $c_1,\cdots,c_s$, $-y_1,\cdots,-y_s$, $(x-1)u_1/2,\cdots,(x-1)u_s/2$ and $m-a-b$ in (6.5.3) and use the well known result

$$P_n^{(b,a)}(-x) = (-)^n P_n^{(a,b)}(x),$$

we get

$$\sum_{n_1,\cdots,n_s=0}^{\infty} \frac{(m-a-b)_{n_1+\cdots+n_s}}{(-c_1-d_1)_{n_1}\cdots(-c_s-d_s)_{n_s}} P_{n_1}^{(c_1-n_1,d_1-n_1)}(y_1)\cdots P_{n_s}^{(c_s-n_s,d_s-n_s)}(y_s)$$

$$\times \left(\frac{1-x}{2}\right)^{n_1+\cdots+n_s} {}_2F_1\left[\begin{matrix}-a-b+m+n_1+\cdots+n_s, -a;\\ -a-b;\end{matrix}\ \frac{2}{x+1}\right] u_1^{n_1}\cdots u_s^{n_s}$$

$$= R_s^{a+b-m} F_A^{(s+1)}\left[m-a-b, -a, -c_1,\cdots,-c_s; -a-b, -c_1-d_1,\cdots,-c_s-d_s;\right.$$

$$2/(x+1)R_s,\ (x-1)u_1/2R_s,\cdots,(x-1)u_s/2R_s\ ]\ ;$$

where $R_s=1-\frac{1}{4}(x-1)\sum_{j=1}^{s}(y_j-1)u_j$; $s=1,2,3,\cdots,$.

Ultimately using the Eulers transformation (6.5.7) and adjusting the parameters on the L.H.S., we get the following multilinear generating relation due to Srivastava and Singhal [124,(30)], which had also been derived by Thakare [128] by a different technique

$$(6.5.11)\quad \sum_{n_1,\cdots,n_s=0}^{\infty}\frac{(m+n_1+\cdots+n_s)!}{(-c_1-d_1)_{n_1}\cdots(-c_s-d_s)_{n_s}}\,u_1^{n_1}\cdots u_s^{n_s}$$

$$\times\ P_{n_1}^{(c_1-n_1,d_1-n_1)}(y_1)\cdots P_{n_s}^{(c_s-n_s,d_s-n_s)}(y_s)$$

$$\times\ P_{m+n_1+\cdots+n_s}^{(a-m-n_1-\cdots-n_s,\ b-m-n_1-\cdots-n_s)}(x)$$

$$=(-)^m(-a-b)_m\left(\frac{x+1}{2}\right)^a\left(\frac{x-1}{2}\right)^{m-a}R_s^{a+b-m}\ F_A^{(s+1)}\ [m-a-b,-a,-c_1,\cdots,-c_s\ ;$$

$$-c_1-d_1,\cdots,-c_s-d_s\ ;\ \frac{2}{(x+1)R_s},\ \frac{x-1}{2R_s}u_1,\cdots,\ \frac{x-1}{2R_s}u_s\ ].$$

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CHAPTER - VII

CERTAIN DUAL SERIES EQUATIONS INVOLVING HAHN POLYNOMIALS IN DISCRETE VARIABLES

7.1 Introduction : In recent years considerable attention has been drawn of several researchers to the solution of problems involving dual equations involving, for instance, trigonometric series, the Fourier-Bessel series, the Dini series and series of Jacobi and Laguerre polynomials. Many of these problems arise in the investigation of certain classes of mixed boundary value problems in potential theory. For a good account of such problems, one can refer to Sneddon [105]. In particular, dual series equations in which the kernels involve Jacobi polynomials of the same indices were first considered by Noble [75] in 1963. Subsequently Srivastava R.P. [126], Dwivedi [38], Thakare [127] also considered dual series equations involving Jacobi polynomials. Lastly Srivastava, H.M. [114, 115] considered problem of determining the unknown sequence $\{A_n\}$ satisfying the general dual series equations

$$(7.1.1) \qquad \sum_{n=0}^{\infty} A_n \frac{\Gamma(u+n+h+1)}{\Gamma(b+n+h+1)} P_{n+h}^{(a,b)}(x) = f(x); \quad -1 \leq x < y,$$

$$(7.1.2) \qquad \sum_{n=0}^{\infty} A_n \frac{\Gamma(v+n+h+1)}{\Gamma(c+n+h+1)} P_{n+h}^{(c,d)}(x) = g(x), \quad y < x \leq 1,$$

where h is an arbitrary non-negative integer, $f(x)$ and $g(x)$ are prescribed functions and in general

$$\min\{a, b, c, d, u, v\} > -1 .$$

To solve these equations H.M. Srivastava applied the technique of Noble, called multiplying factor technique with adequate modifications.

It is interesting to note that the problems dealt with so far had been those involving continuous variables. In the present Chapter we have attempted to deal with a problem involving orthogonal Hahn polynomials in discrete variables, defined by [59] relation

$$(7.1.3) \quad Q_n(x;a,b,N) = {}_3F_2\left[\begin{matrix} -n, & n+a+b+1, & -x\,; \\ & a+1, & -N\,; \end{matrix} \; 1\right].$$

7.2 Statement of the Problems :

Theorem I : Let $\{A_n\}$ be an unknown sequence satisfying the dual series equations :

$$(7.2.1) \quad \sum_{n=0}^{N-h} A_n (u+1)_{n+h}\, Q_{n+h}(x;a,b,N) = f(x); \quad 0 \leqslant x \leqslant y ,$$

$$(7.2.2) \quad \sum_{n=0}^{N-h} A_n \frac{(v+1)_{n+h}\,(c+1)_{n+h}}{(d+1)_{n+h}}\, Q_{n+h}(x;c,d,N) = g(x) ;$$

$$y < x \leqslant N ,$$

where h is an arbitrary non-negative integer, $f(x)$ and $g(x)$ are prescribed functions,

$$(7.2.3)\quad a+b = c+d = u+v$$

$$(7.2.4)\quad v > d-k > -1, \quad u-a+m > 0$$

and in general

$$(7.2.5)\quad \min\{a, b, c, d, u, v\} > -1 .$$

then the unknown sequence $\{A_n\}$ are determined by the relation

$$(7.2.6)\quad A_n = \frac{(-)^{n+h}(-N)_{n+h}(a+b+1)_{n+h}(a+b+2n+2h+1)}{(n+h)!\,(a+b+N+2)_{n+h}(v+1)_{n+h}(a+b+1)}$$

$$\times \binom{N+a+b+1}{N}^{-1} \Big[\sum_{z=0}^{y} \binom{N-z+v}{N-z} Q_{n+h}(z;u,v,N)F(z)$$

$$+ (-)^k \sum_{z=y+1}^{N} \binom{z+u}{z} Q_{n+h}(z;u,v,N)G(z) \Big] ,$$

where

$$(7.2.7)\quad F(z) = \nabla_z^m \Big[\sum_{x=0}^{z} \binom{x+a}{x}\binom{z-x+m+u-a-1}{z-a} f(x) \Big] ,$$

$$(7.2.8)\quad G(z) = \sum_{x=z}^{N} \binom{x-z+v-d+k-1}{x-z} \Delta_x^k \Big[\binom{N-x+d}{N-x} g(x) \Big] ,$$

(7.2.9) $$\nabla_x f(x) = f(x) - f(x-1)$$

and

(7.2.10) $$\Delta_x f(x) = f(x-1) - f(x) .$$

<u>Theorem II</u> : Let $\{A_n\}$ be a known sequence satisfying the dual series equations :

(7.2.11) $$\sum_{n=0}^{N-h} A_n (u+1)_{n+h} Q_{n+h}(x;a,b,N) = \phi(x) ; y < x \leqslant N ,$$

(7.2.12) $$\sum_{n=0}^{N-h} A_n \frac{(v+1)_{n+h} (c+1)_{n+h}}{(d+1)_{n+h}} Q_{n+h}(x;c,d,N) = \Psi(x) ;$$

$$0 \leqslant x \leqslant y ,$$

where the coefficient A_n is given by (7.2.6), h is a non-negative integer and in addition to the parametric constraints given by (7.2.3), (7.2.4) and (7.2.5).

Then the unknown functions $\phi(x)$ and $\Psi(x)$ are given by :

(7.2.13) $$\phi(x) = \binom{x+a}{x}^{-1} \nabla_x^r \Big[\sum_{z=0}^{y} \binom{x-z+a-u+r-1}{x-z} F(z)$$

$$+ (-)^k \sum_{z=y+1}^{x} \binom{z+u}{z} \binom{N-z+v}{N-z} \binom{x-z+a-u+r-1}{x-z} G(z) \Big]$$

$$- \binom{x+a}{x}^{-1} \nabla_x^r \Big[\binom{x+a+r}{x} \sum_{z=0}^{y} \binom{N-z+v}{N-z} M(x,z)\, F(z)$$

$$+ (-)^k \binom{x+a+r}{x} \sum_{z=y+1}^{N} \binom{z+u}{z} M(x,z)\, G(z) \Big] ,$$

where r being a non-negative integer such that

(7.2.14) $\quad a - u + r > 0 \,,\; b - r > -1$

(7.2.15) $\quad M(x,z) = \sum_{n=0}^{h-1} R_n(x,z)$ and

(7.2.16) $$R_n(x,z) = \frac{(-)^n (-N)_n (a+b+1)_n (u+1)_n (a+b+2n+1)}{n!\,(a+b+N+2)_n (v+1)_n (a+b+1)}$$

$$\times \binom{N+a+b+1}{N}^{-1} Q_n(x;a+r,b-r,N)\, Q_n(z;u,v,N)$$

and

(7.2.17) $$\Psi(x) = \binom{N-x+d}{N-x}^{-1} (-\Delta_x)^s \Big[\sum_{z=x}^{y} \binom{N-z+v}{N-z} \binom{z+u}{z}^{-1}$$

$$\times \binom{z-x+u-c+s-1}{z-x} F(z) + (-)^k \sum_{z=y}^{N} \binom{z-x+u-c+s-1}{z-x} G(z) \Big]$$

$$- \binom{N-x+d}{N-x}^{-1} (-\Delta_x)^s \Big[\binom{N-x+d+s}{N-x} \sum_{z=0}^{y} \binom{N-z+v}{N-z} K(x,z)\, F(z)$$

$$+(-)^k \binom{N-x+d+s}{N-x} \sum_{z=y}^{N} \binom{z+u}{z} K(x,z)\ G(z) \Big] ,$$

where s is a non-negative integer such that

$$(7.2.18) \quad u > c - s > -1 ,$$

$$(7.2.19) \quad K(x,z) = \sum_{n=0}^{h-1} S_n(x,z) \quad \text{and}$$

$$(7.2.20) \quad S_n(x,z) = \frac{(-)^n\ (-N)_n\ (c+d+1)_n\ (c-s+1)_n\ (c+d+2n+1)}{n!\ (c+d+N+2)_n\ (d+s+1)_n\ (c+d+1)}$$

$$\times \binom{N+a+b+1}{N}^{-1} Q_n(x;c-s,d+s,N)\ Q_n(z;u,v,N) .$$

7.3 Preliminary Results. For the solution of our problems of section 7.2, the following results involving Hahn polynomials will be required:

(i) The following convenient forms of the orthogonality properties of the Hahn polynomials given by Karlin and McGregar [59] :

$$(7.3.1) \quad \sum_{x=0}^{N} \binom{x+a}{x} \binom{N-x+b}{N-x} Q_n(x;a,b,N)\ Q_m(x;a,b,N)$$

$$= \frac{(-)^n\ n!\,(N+a+b+2)_n\ (b+1)_n\ (a+b+1)}{(-N)_n (a+1)_n\ (a+b+1)_n\ (2n+a+b+1)} \binom{N+a+b+1}{N} \delta_{mn} ,$$

where $0 \leqslant m,\ n \leqslant N$

and

$$(7.3.2)\quad \sum_{n=0}^{N} \frac{(-)^n\,(-N)_n\,(a+1)_n\,(a+b+1)_n\,(2n+a+b+1)}{n!\,(N+a+b+2)_n\,(b+1)_n\,(a+b+1)}$$

$$\times\ Q_n(x;a,b,N)\,Q_m(y;a,b,N)$$

$$= \binom{x+a}{x}^{-1}\binom{N-x+b}{N-x}^{-1}\binom{N+a+b+1}{N}\delta_{xy}\,,$$

where x, y are integers and $0 \leqslant x,\ y \leqslant N$.

(ii) For $p > 0$ and $a > -1$, we have

$$(7.3.3)\quad \sum_{x=0}^{z}\binom{x+a}{x}\binom{z-x+p-1}{z-x}Q_n(x;a,b,N)$$

$$= \binom{z+a+p}{z}\,Q_n(z;a+p,b-p,N)\ ,$$

which is the slightly modified form of the following summation formula given by Gasper [45,(2.1)]

$$Q_n(z;a+p,b-p,N) = \sum_{x=0}^{z}\binom{z}{x}\frac{(a+1)_x\,(p)_{z-x}}{(a+p+1)_z}\,Q_n(x;a,b,N).$$

By use of the transformation (3.5.6), we can write equation (7.1.3) as

$$(7.3.4)\quad Q_n(x;a,b,N) = \frac{(-)^n(b+1)_n}{(a+1)_n}\ {}_3F_2\left[\begin{matrix}-n, n+a+b+1, & -N+x\ ; \\ b+1, & -N\ ;\end{matrix}\ 1\right].$$

Hence we get the following relation

$$(7.3.5) \quad Q_n(x;a,b,N) = \frac{(-)^n (b+1)_n}{(a+1)_n} Q_n(N-x;b,a,N)$$

On replacing a,b,x, and z respectively by b,a,N-x and N-z and using the relation (7.3.5) in equation (7.3.3), gives its following complementary result.

$$(7.3.6) \quad \sum_{x=z}^{N} \binom{N-x+b}{N-x} \binom{x-z+p-1}{x-z} Q_n(x;a,b,N)$$

$$= \frac{(a-p+1)_n \ (b+1)_n}{(b+p+1)_n \ (a+1)_n} \binom{N-z+b+p}{N-z} Q_n(z;a-p,b+p,N),$$

for $p > 0$ and $b > -1$.

(iii) In our analysis we shall also use the following two difference formulas involving Hahn polynomials :

$$(7.3.7) \quad \nabla_x^m \left[\binom{x+a+m}{x} Q_n(x;a+m,b-m,N) \right]$$

$$= \binom{x+a}{x} Q_n(x;a,b,N) \ ;$$

for non-negative integer m and $a > -1$,

and

$$(7.3.8) \quad \Delta_x^m \left[\binom{N-x+b+m}{N} Q_n(x;\ a-m,\ b+m,\ N) \right]$$

$$= \frac{(-)^m \ (b+m+1)_n \ (a+1)_n}{(a-m+1)_n \ (b+1)_n} \binom{N-x+b}{N-x} Q_n(x;\ a,\ b,\ N) \ ;$$

for $b > -1$ and integer $m \geqslant 0$.

From equations (7.1.3) and (7.3.4), we see that the above results are special cases of

$$(7.3.9)\quad \nabla_x^m \left[\binom{x-u+a+m}{x-u} {}_{E+1}F_{G+1} \left[\begin{matrix} -x+u\ , & (e)\ ; \\ a+1+m\ , & (g)\ ; \end{matrix}\ t \right] \right]$$

$$= \binom{x-u+a}{x-u} {}_{E+1}F_{G+1} \left[\begin{matrix} -x+u\ , & (e)\ ; \\ a+1\ , & (g)\ ; \end{matrix}\ t \right]$$

(when $u=0$, $E=2$, $G=1$, $e_1=n+a+b+1$, $e_2= -n$, $g_1= -N$ and $t=1$)

and

$$(7.3.10)\quad \Delta_x^m \left[\binom{u-x+b+m}{u-x} {}_{E+1}F_{G+1} \left[\begin{matrix} -u+x\ , & (e)\ ; \\ b+1+m\ , & (g)\ ; \end{matrix}\ t \right] \right]$$

$$= (-)^m \binom{u-x+b}{u-x} {}_{E+1}F_{G+1} \left[\begin{matrix} -u+x\ , & (e)\ ; \\ b+1\ , & (g)\ ; \end{matrix}\ t \right]$$

(when $u=N$, $E=2$, $G=1$, $e_1=n+a+b+1$, $e_2= -n$, $g_1= -N$ and $t=1$)

respectively.

To prove (7.3.9) consider

$$\nabla_x \left[\binom{x-u+a+m}{x-u} {}_{E+1}F_{G+1} \left[\begin{matrix} -x+u\ , & (e)\ ; \\ a+1+m, & (g)\ ; \end{matrix}\ t \right] \right]$$

$$= \nabla_x \left[\sum_{r=0}^{\infty} \frac{\Gamma(x-u+a+1+m)}{\Gamma(a+1+m+r)\,\Gamma(x-u+1)} \cdot \frac{(-x+u)_r\ ((e))_r}{((g))_r\ r!}\, t^r \right]$$

$$= \sum_{r=0}^{\infty} \frac{((e))_r}{\Gamma(a+1+m+r)\,((g))_r\, r!}\, t^r$$

$$\times \left[\frac{\Gamma(x-u+a+1+m)\,(-x+u)_r}{\Gamma(x-u+1)} - \frac{\Gamma(x-u+a+m)\,(-x+u+1)_r}{\Gamma(x-u)} \right.$$

$$= \sum_{r=0}^{\infty} \frac{\Gamma(x-u+a+m)}{\Gamma(a+m+r)\,\Gamma(x-u+1)} \cdot \frac{(-x+u)_r\,((e))_r}{((g))_r\, r!}\, t^r$$

$$= \binom{x-u+a+m-1}{x-u}\ {}_{E+1}F_{G+1} \left[\begin{matrix} -x+u, & (e) & ; \\ a+m, & (g) & ; \end{matrix}\ t \right].$$

Hence by iteration we get the required result (7.3.9).

In a similar manner we can easily obtain (7.3.10).

7.4 Proof of the Theorem I : To our series equation (7.2.1) multiply with $\binom{x+a}{x}\binom{z-x+m+p-1}{z-x}$, for some suitable p and non-negative integer m, then summing the series from $x=0$ to $x=z$ on both sides, and using the summation formula (7.3.3), we get

$$(7.4.1) \quad \sum_{n=0}^{N-h} A_n (u+1)_{n+h} \binom{z+a+m+p}{z} Q_{n+h}(z; a+m+p, b-m-p, N)$$

$$= \sum_{x=0}^{z} \binom{x+a}{x} \binom{z-x+m+p-1}{z-x} f(x)\ ,$$

where $0 \leqslant z \leqslant y$, $a > -1$ and $p+m > 0$.

On operating both sides of this last equation (7.4.1) by ∇_z , m times and using the difference formula (7.3.7), we have

$$(7.4.2)\quad \sum_{n=0}^{N-h} A_n (u+1)_{n+h} \binom{z+a+p}{z} Q_{n+h}(z;a+p,b-p,N)$$

$$= \nabla_z^m \Big[\sum_{x=0}^{z} \binom{x+a}{x} \binom{z-x+m+p-1}{z-x} f(x) \Big] ,$$

where $0 \leqslant z \leqslant y$, $a > -1$, $p+m > 0$ and $a+p > -1$.

Next we multiply our series equation (7.2.2) by $\binom{N-x+d}{N-x}$ and operate both sides by Δ_x , k times for a non-negative integer k, using the difference formula (7.3.8), we thus obtain

$$(7.4.3)\quad \sum_{n=0}^{N-h} A_n \frac{(v+1)_{n+h}\,(c+k+1)_{n+h}}{(d-k+1)_{n+h}} \binom{N-x+d-k}{N-x} Q_{n+h}(x;c+k,d-k,N)$$

$$= (-)^k \Delta_x^k \Big[\binom{N-x+d}{N-x} g(x)\Big] ,$$

where $y < x \leqslant N$ and $d-k > -1$.

For a suitable constant q, multiply to equation (7.4.3) by $\binom{x-z+q+k-1}{x-z}$, then summing the series from $x=z$

to x=N on both sides, using the summation formula (7.3.6), we get

$$(7.4.4)\quad \sum_{n=0}^{N-h} A_n (v+1)_{n+h} \frac{(c-q+1)_{n+h}}{(d+q+1)_{n+h}} \binom{N-z+d+q}{N-z}$$

$$\times\ Q_{n+h}(z;c-q,d+q,N)$$

$$= \sum_{x=z}^{N} \binom{x-z+q+k-1}{x-z} (-\Delta_x)^k \left[\binom{N-x+d}{N-x} g(x)\right],$$

where $y < z \leqslant N$, $d-k > -1$, $q+k > 0$ and integer $k \geqslant 0$.

Now under the parametric condition (7.2.3), if we choose p and q such that

$$a+p = c-q = u \quad \text{and} \quad b-p = d+q = v.$$

then equations (7.4.2) and (7.4.4) can be written as

$$(7.4.5)\quad \sum_{n=0}^{N-h} A_n (u+1)_{n+h} \binom{z+u}{z} Q_{n+h}(z;u,v,N) = F(z)$$

where $0 \leqslant z \leqslant y$, $a > -1$, $u > -1$ and $F(z)$ is given by (7.2.7), and

$$(7.4.6)\quad \sum_{n=0}^{N-h} A_n (u+1)_{n+h} \binom{N-z+v}{N-z} Q_{n+h}(z;u,v,N) = (-)^k\ G(z),$$

where $y < z \leqslant N$, $d-k > -1$, $v > -1$ and $G(z)$ is given by (7.2.8).

Now, multiplying equation (7.4.5) by

$\binom{N-z+v}{N-z} Q_j(z;u,v,N)$, and summing the series from

$z=0$ to y on both sides, we have

$$(7.4.7)\quad \sum_{n=0}^{N-h} A_n (u+1)_{n+h} \sum_{z=0}^{y} \binom{z+u}{z} \binom{N-z+v}{N-z}$$

$$\times\ Q_{n+h}(z;u,v,N)\ Q_j(z;u,v,N)$$

$$= \sum_{z=0}^{y} \binom{N-z+v}{N-z} Q_j(z;u,v,N)\ F(z)$$

Next if we multiply equation (7.4.6) by

$\binom{z+u}{z} Q_j(z;u,v,N)$ and summing the series from $z=y+1$ to N,

we get

$$(7.4.8)\quad \sum_{n=0}^{N-h} A_n (u+1)_{n+h} \sum_{z=y+1}^{N} \binom{z+u}{z} \binom{N-z+v}{N-z}$$

$$\times\ Q_{n+h}(z;u,v,N)\ Q_j(z;u,v,N)$$

$$= (-)^k \sum_{z=y+1}^{N} \binom{z+u}{z} Q_j(z;u,v,N)\ G(z).$$

On adding (7.4.7) and (7.4.8), we obtain

$$(7.4.9)\quad \sum_{n=0}^{N-h} A_n (u+1)_{n+h} \sum_{z=0}^{N} \binom{z+u}{z} \binom{N-z+v}{N-z}$$

$$\times\, Q_{n+h}(z;u,v,N)\; Q_j(z;u,v,N)$$

$$= \sum_{z=0}^{y} \binom{N-z+v}{N-z} Q_j(z;u,v,N)\; F(z)$$

$$+(-)^k \sum_{z=y+1}^{N} \binom{z+u}{z} Q_j(z;u,v,N)\; G(z)\ ,$$

which with the help of orthogonality properly (7.3.1), gives required result (7.2.6) under the parametric conditions (7.2.3), (7.2.4) and (7.2.5). Hence theorem I is proved.

7.5 Proof of The Theorem II. For non-negative integer r, we can easily change equation (7.2.11), by applying the difference formula (7.3.7) into the form

$$(7.5.1)\quad \phi(x) = \binom{x+a}{x}^{-1} \nabla_x^{r} \Big[\binom{x+a+r}{x}$$

$$\times \sum_{n=0}^{N-h} A_n (u+1)_{n+h}\; Q_{n+h}(x;a+r,b-r,N) \Big]\ ,$$

where $y < x \leqslant N$.

Now substituting the coefficients A_n from (7.2.6) into the above expression (7.5.1), we have

$$(7.5.2)\qquad \phi(x) = \binom{x+a}{x}^{-1} \nabla_x^r \Big\{ \binom{x+a+r}{x}$$

$$\times \sum_{n=0}^{N-h} \frac{(-)^{n+h}(-N)_{n+h}\,(a+b+1)_{n+h}\,(a+b+2n+2h+1)}{(n+h)!\,(a+b+N+2)_{n+h}\,(v+1)_{n+h}\,(a+b+1)}$$

$$\times\ (u+1)_{n+h} \binom{N+a+b+1}{N}^{-1} \Big[\sum_{z=0}^{y} \binom{N-z+v}{N-z} Q_{n+h}(z;u,v,N)\ F(z)$$

$$+(-)^k \sum_{z=y+1}^{N} \binom{z+u}{z} Q_{n+h}(z;u,v,N)\,G(z)\Big]\, Q_{n+h}(x;a+r,b-r,N)\Big\}$$

$$= \binom{x+a}{x}^{-1} \nabla_x^r \Big[\binom{x+a+r}{x} \sum_{z=0}^{y} \binom{N-z+v}{N-z}$$

$$\times \sum_{n=0}^{N-h} R_{n+h}(x,z)\ F(z)\ +(-)^k \binom{x+a+r}{x}$$

$$\times \sum_{z=y+1}^{N} \binom{z+u}{z} \sum_{n=0}^{N-h} R_{n+h}(x,z)\ G(z)\ \Big]$$

$$= \binom{x+a}{x}^{-1} \nabla_x^r \Big[\binom{x+a+r}{x} \sum_{z=0}^{y} \binom{N-z+v}{N-z} \sum_{n=0}^{N} R_n(x,z)\ F(z)$$

$$+\ (-)^k \binom{x+a+r}{x} \sum_{z=y+1}^{N} \binom{z+u}{z} \sum_{n=0}^{N} R_n(x,z)\ G(z)\ \Big]$$

$$-\ \binom{x+a}{x}^{-1} \nabla_x^r \Big[\binom{x+a+r}{x} \sum_{z=0}^{y} \binom{N-z+v}{N-z} M(x,z)\ F(z)$$

$$-\ (-)^k \binom{x+a+r}{x} \sum_{z=y+1}^{N} \binom{z+u}{z} M(x,z)\ G(z)\ \Big]\ ,$$

where $M(x,z)$ and $R_n(x,z)$ are given by equations (7.2.15) and (7.2.16) respectively.

From the summation formula (7.3.6) and parametric condition (7.2.3), equation (7.2.16) can be written as

$$(7.5.3)\quad R_n(x,z) = \binom{N-z+v}{N-z}^{-1} \sum_{w=z}^{N} \binom{N-w+b-r}{N-w} \binom{w-z+a+r-u-1}{w-z}$$

$$X \binom{N+a+b+1}{N}^{-1} \frac{(-)^n\,(-N)_n\,(a+r+1)_n\,(a+b+1)_n\,(2n+a+b+1)}{n!\,(N+a+b+2)_n\,(b-r+1)_n\,(a+b+1)}$$

$$X\ Q_n(w;a+r,b-r,N)\ Q_n(x;a+r,b-r,N)\ ,$$

where $b-r > -1$ and $a-u+r > 0$.

Using the ortegonality property (7.3.2), the equation (7.5.3) yields

$$(7.5.4)\quad \sum_{n=0}^{N} R_n(x,z) = \binom{N-z+v}{N-z}^{-1} \binom{x+a+r}{x}^{-1} \binom{x-z+a-u+r-1}{x-z} H(x-z)\ ,$$

where $b-r > -1$, $a-u+r > 0$ and $H(t)$ denotes Heavisides unit step function, defined as

$$H(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geq 0\ . \end{cases}$$

On substituting the value $\sum_{n=0}^{N} R_n(x,z)$ from equation (7.5.4) into (7.5.2), we finally get required result (7.2.13).

Next from the difference formula (7.3.8) and our dual series equation (7.2.12), we have

$$(7.5.5)\quad \Psi(x) = \binom{N-x+d}{N-x}^{-1} (-\Delta_x)^s \Big[\binom{N-x+d+s}{N-x}$$

$$\times \sum_{n=0}^{N-h} A_n \frac{(v+1)_{n+h}(c-s+1)_{n+h}}{(d+s+1)_{n+h}} Q_{n+h}(x; c-s, d+s, N)$$

which on substituting the coefficients A_n from (7.2.6), gives us

$$(7.5.6)\quad \Psi(x) = \binom{N-x+d}{N-x}^{-1}$$

$$\times (-\Delta_x)^s \Big[\binom{N-x+d+s}{N-x} \sum_{z=0}^{y} \binom{N-z+v}{N-z} \sum_{n=0}^{N} S_n(x,z)\ F(z)$$

$$+ (-)^k \binom{N-x+d+s}{N-x} \sum_{z=y+1}^{N} \binom{z+u}{z} \sum_{n=0}^{N} S_n(x,z)\ G(z) \Big]$$

$$- \binom{N-x+d}{N-x}^{-1} (-\Delta_x)^s \Big[\binom{N-x+d+s}{N-x} \sum_{z=0}^{y} \binom{N-z+v}{N-z} K(x,z)\ F(z)$$

$$+ (-)^k \binom{N-x+d+s}{N-x} \sum_{z=y+1}^{N} \binom{z+u}{z} K(x,z)\ G(z) \Big] ,$$

where $S_n(x,z)$ and $K(x,z)$ given by (7.2.20) and (7.2.19) respectively.

From the summation formula (7.3.3) and equation (7.2.20), we have

$$(7.5.7) \quad \sum_{n=0}^{N} S_n(x,z) = \sum_{w=0}^{z} \binom{z+u}{z}^{-1} \binom{w+c-s}{w} \binom{z-w+u+s-c-1}{z-w}$$

$$\times \sum_{n=0}^{N} \frac{(-)^n \, (-N)_n \, (c+d+1)_n \, (c-s+1)_n \, (c+d+2n+1)}{n! \, (c+d+N+2)_n \, (d+s+1)_n \, (c+d+1)}$$

$$\times \binom{N+c+d+1}{N}^{-1} Q_n(x;c-s,d+s,N) \; Q_n(w;c-s,d+s,N)$$

where $u > c-s > -1$.

By use of orthogonality property (7.3.2), we have

$$\sum_{n=0}^{N} S_n(x,z) = \binom{z+u}{z}^{-1} \binom{z-x+u+s-c-1}{z-x} \binom{N-x+d+s}{N-x}^{-1} \times H(z-x) \; ,$$

which on putting in equation (7.5.6), gives required result (7.2.17). Thus theorem II is proved.

<u>Remark</u> : From equation (7.1.3), it can be easily verified that

$$\lim_{N \to \infty} Q_n(xN \; ; \; a,b,N) = \frac{n!}{(1+a)_n} P_n^{(a,b)}(1-2x)$$

and

$$\lim_{N \to \infty} Q_n(x; \; a-1, (1-b)N/c, \; N) = M_n(x;a,b) \; ,$$

where $M_n(x;a,b)$ is called Mexiner polynomials defined as (see [119, 1.9(3)])

$$M_n(x;b,c) = {}_2F_1(-n,-x;\ b\ ;\ 1-c^{-1})$$

and the orthogonality property relation of Mexiner polynomials is

$$\sum_{x=0}^{\infty} M_n(x;a,b)\ M_m(x;a,b)\ \frac{(a)_x}{x!}\, b^x = 0$$

$$\text{for}\quad m \neq n;\ 0 < b < 1;\quad a > 0$$

Hence, our dual series equations (7.2.1) and (7.2.2), on replacing x by $(1-x)N/2$, taking $\text{Lim}\ N \to \infty$ and adjusting the parameters reduces into Srivastava equations (7.1.1) and (7.1.2) respectively involving Jacobi polynomials of continuous variables.

Further, if we replace a, b, c, d respectively by $a-1$, $(1-b)N/b$, $c-1$, $(1-d)N/d$ and taking $\text{Lim}\ N \to \infty$ in (7.2.1) and (7.2.2), we get the following dual series equations involving Mexiner polynomials

$$\sum_{n=0}^{\infty} A_n (u+1)_{n+h}\ M_{n+h}(x;a,b) = f(x);\ 0 \leq x \leq Y$$

and

$$\sum_{n=0}^{\infty} A_n \left(\frac{dv}{1-d}\right)^{n+h} (c)_{n+h}\ M_{n+h}(x;c,d) = g(x);\ Y < x \leq \infty .$$

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BIBLIOGRAPHY

1. AGRAWAL, B.M. (1967) : "Operational formulae for classical polynomials" Vij. Par. Anu. Pat., 43-50.

2. AGRAWAL, B.M. (1973) : "Transformation of hypergeometric series" Vij. Pard. Anu. Pat., 16, 3, 169-176.

3. AGRAWAL, B.M. & AGRAWAL, H.C. (1975) : "An application of the difference operator in the study of some polynomials" J. Pure Appl. Math., 6(4), 456-462.

4. AGRAWAL, B.M. & MANGLI~~C~~, R.C. (1974) : "Finite summations involving hypergeometric functions", Vij. Pard. Anu. Pat., 17, 3, 197-200.

5. AGRAWAL, H.C. (1975) : "A note on generalized Tricomi polynomials", Nouvelle serce, tome, 19(33), 5-8.

AGRAWAL, H.C. (1978) : "On hypergeometric function of three variables", Vij. Par. Anu. Pat., 21, 133-138.

7. AGRAWAL, H.C. (1981) : "Expansions involving hypergeometric functions of two and three variables", Bull. Tech. Univ. Istanbul, 34, No. 1, 19-28.

8. AGRAWAL, R.P. (1953) : "Associated basic hypergeometric series", Proc. Glasgow Math. Soc. Assoc., 1, 182-184.

9. AGRAWAL, R.P. (1954) : "Some relations between basic hypergeometric functions of two variables", Rend. Circ. Mat. Polermo, 3, 1-7.

10. AL-SALAM, W.A. (1957) : "The Bessel polynomials", Duke Math. J., 24, 529-545.

11. AL-SALAM, W.A. (1964) : "Operational representation for Laguerre and other polynomials", Duke Math. J., 31, 127-142.

12. APPELL, P. (1880) : "Sur une classe de polynomes", Ann. Sci. Ecole. Norm. Sup., 9, 119-144.

13. APPELL, P. (1880) : "Sur les series Hypergeometriqués de Duex variables et sur des Equations Defferentielles Lineaires aur Derivees Partielles", C.R. Acad. Sci. Paris, 19, 296-731.

14. APPELL, P. (1901) : Sur une suite de polynomes ayant toutes leurs Racinesreeles", Archiv. der Math. und Phy., pp. 69.

15. APPELL, P. & KAMPE DE FERIET, J. (1926) : "Functions Hypergeometriques et Hypersphériqués : polynomes d'Hermite", Gauthier-Villars, Paris.

15a. ASKEY, R. & WILSON, J. (1985) : "Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials", Ame. Math. Soc., Vol. 54, No. 319, 1-55.

16. BAILEY, W.N. (1929) : "Some identities involving generalized hypergeometric series", Proc. London Math. Soc.(2), 29, 503-516.

17. BAILEY, W.N. (1935) : "Generalized hypergeometric series", Cambridge Univ. Press, Cambridge; reprint, Cambridge Tracts in math and Math. Phys., No. 32, Stechert-Hafner, New York, 1964, MR 32 # 2625.

18. BOAS, R.P. Jr. & BUCK, R.C. (1956) : "Polynomials defined by generating relations", Amer. Math. Monthly, 63, 626-632.

19. BRAFMAN, F. (1957) : "Some generating functions of Laguerre and Hermite polynomials", Cand. J. Math, 9, 180-187.

20. BRENKE, W.C. (1945) : "On generating functions of polynomial systems", Amer. Math. Monthly, 52, 297-301.

21. BURCHNALL, J.L. (1941) : "A note on the polynomial of Hermite", Quart. J. Math. Oxford Ser., 12, 9-11.

22. BURCHNALL, J.L. & CHAUNDY, T.W. (1941) : "Expansions of Appell's double hypergeometric functions II". Quart. J. Math. Oxford Ser., 12, 112-128, MR 3, 118.

23. CARLITZ, L. (1956) : "On Jacobi polynomials", Bell. Un. Mat. Stal., 11, 371-381.

24. CARLITZ, L. (1958) : "On some polynomials of Tricomi", Boll. V.M.I. (3), 13, 58-64.

25. CARLITZ, L. (1960) : "A note on Laguerre polynomials", Michigan Math. J., 7, 219-223.

26. CARLITZ, L. (1963) : "A Saalschitzian theorem for double series", Journal London Math. Soc., 38, 415-418.

27. CARLITZ, L. (1964) : "Another Saalschiitzian theorem for double series", Rend. Semi. Mat. Univ. Paodova, 34, 200-203.

28. CARLITZ, L. (1967) : "A summation theorem for double hypergeometric series", Rend. Semi. Mat. Univ. Paodova, 37, 230-233.

29. CARLITZ, L. (1967) : "A transformation for multiple hypergeometric series", Monatsh Math., 71, 1-6, MR 34 # 6162.

30. CARLITZ, L. (1967) : "Summation of a double hypergeometric series", Matematiche (Catania), 22, 138-142, MR 35 # 5665.

31. CARLITZ, L. & SRIVASTAVA, H.M. (1976) : "Some hypergeometric polynomials associated with the Lauricella function F_D of several variables. I and II", Mat. Vesnik, 13(28), 41-47 and 143-152.

32. CHAK, A.N. (1956) : "A class of polynomials and a generalization of Stirling numbers", Duke Math. J., 23, 45-55.

33. CHATTERJEA, S.K. (1964) : "A generalization of the Bessel polynomials", Mathematica, 29, 19-29.

34. CHATERJEA, S.K. (1966) : "Some operational formulas connected with a function defined by a generalized Rodrigues' formula", Acta Math. Acad. Sci. Hungar., 17, 379-385.

35. CHAUNDY, T.W. (1943) : "An extension of the hypergeometric function", Quart. J. Math., 14, 55-78.

36. DESHPANDE, V.L. & BHISE, V.M. (1970) : "A generating function for the Rice's polynomial", Math. Bechnk, 7(22), 169-172.

37. DIDON, F. (1870) : "Developpments sur certaines series de polynomes a un nombre quelonque de variables", Ann. Sci. Ecals Norm. Sup., 7, 247-292.

38. DWIVEDI, A.P. (1968) : "Certain dual series equations involving Jacobi polynomials", J. Indian Math. Soc. (N.S.), 32, Suppl. I, 287-300.

39. ERDELYI, A. (1934) : "Beitrag zur theorie der konfluenten hypergeometrischen funktionen von mehreren veranderlichen, S-B", Akad. Wiss. Wien Abt. IIa, Math. Natur. K 1.146, 431-467.

40. ERDELYI, A., MAGNUS, W., OBERHETTINGER, F., TRICOMI, F.G., et al (1953) : Higher Transcendental Functions", Vol. I, McGraw-Hill Book Co., Inc.,

41. ERDELYI, A., MAGNUS, W., OBERHETTINGER, F. & TRICOMI, F.G. (1953) : "Higher Transcendental Functions", Vol. II, McGraw-Hill Book Co., Inc.

42. ERDEL~~H~~I, A. et al. (1955) : "Higher Transcendental Functions", Vol. 3, Mc-Graw-Hill, New York. y

43. EXTON, H. (1976) : "Multiple Hypergeometric Functions And Applications", Ellis Harwood Limited England (London).

44. EXTON, H. (1983) : "q-Hypergeometric Functions And Applications", Ellis Harwood Limited, England.

45. GASPER, G. (1974) : "Projection formula for orthogonal & polynomials of a discrete variable", J. Math. Anal. Appl., 45, 176-198.

46. GAUSS, C.F. (1866) : "Werke, Vol.3", Konigliche Gesellschaft der wissenschaften, Gottingen.

47. GOULD, H.W. (1964) : "The operator $(a^{x} \Delta)^{n}$ and Stirling numbers of the first kind" The American Math. Monthly, 71, 850-858.

48. GREEN, G. (1834) : "On the determination of the exterior and interior attractions of ellipsoids of variable densities", Trans. Cambridge Phil. Soc., 5, 395-423.

49. HEINE, E. (1878) : "Handbuch der Theorie der Kugel functionen", Vol. I, Berlin, 97-125.

50. HERMITE, C. (1865) : "Sur quelques developpments en sirie de functions de plusieurs variables" C.R. Acad. Sci. Paris, 60, 370, 432, 461 and 512.

51. HORN, J. (1931) : "Hypergeometric Funktionen Zioeier Veranderlichen", Math. Ann., 105, 381-407.

52. HUFF, W.N. (1947) : "The type of polynomials generated by $f(xt)\phi(t)$", Duke Math. J., 14, 1091-1104.

53. HUMBERT, P. (1920) : "The confluent hypergeometric functions of two variables", Proc. Roy. Soc. Edinburgh Sect. A., 41, 73-82.

54. JACKSON, F.H. (1951) : "Basic Integration", Quart. J. Math. Oxford Ser., 2, 1-16.

55. JAIN, R.N. (1966) : "Sum of a double hypergeometric series", Matematiche (Catania), 21, 300-301.

56. JOSHI, C.M. & PRAJAPAT, M.L. (1977) : "On some properties of a class of polynomials unifying the generalized Hermite, Laguerre and Bessel polynomials", The Math. Stud. Vol. 45, No. 2, 74-86.

57. JOSHI, C.M. & PRAJAPAT, M.L. (1981) : "The operator $T_{k,q}$ and characterization of a class of polynomials by the generalized Rodrigues' formula", Kyungpook Math. J., Vol. 21, No. 1, 135-145.

58. KAMPE DE FERIET, J. (1921) : "Les Fonctions Hypergeometriques d' Order Superieur a Deux Variables", C.R. Acad. Sci. Paris Ser. A-B, 173-489.

59. KARLIN, S. & McGREGOR, J.L. (1961) : "The Hahn polynomials, formulas and applications", Scripta Math., 26, 33-46.

60. KHANDEKAR, P.R. (1964) : "On a generalization of Rice's polynomial", Proc. Nat. Acad. Sci. India, 34, 157-162.

61. KONHAUSER, J.D.E. (1967) : "Biorthogonal polynomials suggested by the Laguerre polynomials" Pacific J. Math., 21, 303-314, MR 35 # 5674.

62. LAPLACE, P.S. de (1812) : "Theorie Analytique des Probabilites", Paris.

63. LAURICELLA, G. (1893) "Sulle funzioni ipergeometriche a piu variable", Rend. Circ. Mat. Palermo, 7, 111-158.

64. MADHEKAR, H.C. & THAKARE, N.K. "Biorthogonal polynomials suggested by the Jacobi polynomials" Pacific J. Math., 100, 417-424.

65. MADHEKAR, H.C. & THAKARE, N.K. (1982) : "Multilinear generating functions for Jacobi polynomials and for their two variable generalizations", Indian J. Pure Appl. Math., 13, 711-716.

66. MADHEKAR, H.C. & THAKARE, N.K. (1982) : "Multilateral generating functions for the products of hypergeometric polynomials and several generalized hypergeometric functions", Kyungpook Math. J. 22, 79-88.

67. MANOCHA, H.L. (1967) : "Some bilinear generating functions for Jacobi polynomials", Proc. Camp. Phil. Soc., 63, 457-459.

68. MANOCHA, H.L. (1969) : "On Laguerre polynomials", Pub. Inst. Math., 9(23), 225-234.

69. MANOCHA, H.L. (1969) : "Bilateral generating functions involving Jacobi polynomials", Proc. Camp. Phil. Soc. 66, 105-107.

70. MANOCHA, H.L. & SHARMA, B.L. (1966) : "Some formulae for Jacobi polynomials", Proc. Camp. Phil. Soc., 62, 459-462.

71. MITTAL, H.B. (1972) : "Polynomials defined by generating relations", Trans. Amer. Math. Soc., 168, 73-84.

72. MITTAL, H.B. (1974) : "Unusual generating relations for polynomial sets", J. Reine Angew. Math., 271, 122-138.

73. MITTAL, H.B. (1974) : "A generalization of the Laguerre polynomials", Czechoslovak. Math. J., 24, 341-348.

74. MITTAL, H.B. (1977) : "Bilinear and bilateral generating relations", Amer. J. Math., 99, 23-55.

75. NOBLE, B. (1963) : "Some dual series equations involving Jacobi polynomials", Proc. Cambridge Philos. Soc., 59, 363-371.

76. PALAS, F.J. (1955) : "The polynomials generated by f(t) exp [p(x)u(t)]" Oklahoma Thesis.

77. PANDEY, R.C. & SARAN, S. (1963) : "On the hypergeometric functions of higher order in two variables" Proc. Rajasthan Acad. Sci., 10, 3-13, MR 30 # 3999.

78. PARASHAR, B.P. (1983) : "The difference operators and external Laguerre polynomials", Bull. Austral. Math. Soc., 28, 111-119.

79. PARASHAR, B.P. (1985) : "On biorthogonal polynomials suggested by the Jacobi polynomials" Indian J. Pure Appl. Math., 16(10), 1132-1137.

80. PATIL, K.R. & THAKARE, N.K. (1975) : "Operational formula for a function defined by a generalized Rodrigues' formula-II", Science J. Shivaji Univ. 15, 1-10.

81. PATIL, K.R. & THAKARE, N.K. (1976) : "New operational formulas and generated functions for Laguerre polynomials" Indian J. Pure App. Math., Vol. 7, No. 10, 1104-1118.

82. PATIL, K.R. & THAKARE, N.K. (1977) : "Bilateral generating function for a function defined by generalized Rodrigues' formula" Indian J. Pure App. Math., Vol. 8, No. 4, 425-429.

83. PRABHAKAR, T.R. & REKHA SUMAN, (1972) : "On a general class of polynomials suggested by Laguerre polynomials" Math. Student, 40, 311-317, MR 53 # 3382.

84. RAINVILLE, E.D. (1945) : "Certain generating functions and associated polynomials" Amer. Math. Monthly, 52, 239-250.

85. RAINVILLE, E.D. (1960) : "Special Functions" Macmillan Co. New York.

86. ROTA, G.C., KAHANER, D. & ODLYZKO, A. (1973) : "Finite operator calculus", J. Math. Anal Appl., 42, 685-760.

87. SAXENA, R.K. (1969) : "A new generating function for Jacobi polynomials", Proc. Cambridge Philos. Soc., 66, 345-347.

88. SHAH, M. (1972) : "A result on Kampe de Feriet function" Glasnik Ser. III 7, 27, 53-56.

89. SHARMA, B.L. (1976) : "Some relations between hypergeometric series", Simon Stevin, 50, 103-110.

90. SHARMA, B.L. (1976) : "Whipple's formulae for double series", Simon Stevin, 50, 183-188.

91. SHARMA, B.L. & ABIODUN, R.F.A. (1973) : "Formulae for hypergeometric functions of two variables", An. Univ. Bucuresti Mat. Mec., 22, 137-139, MR 50 # 10347.

92. SHEFFER, I.M. (1939) : "Some properties of polynomial sets of type zero", Duke Math. J., 5, 590-622.

93. SHIVEHLY, R.L. "On pseddo Laguerre polynomials", Ph.D. thesis Univ. of Michigan.

94. SHRIVASTAVA, P.N. (1969) : "Classical polynomials - a unified presentation", Publ. Inst. Math.(Beograd) (N.S.), 23(37), 169-177.

95. SHRIVASTAVA, P.N. (1969) : "On the polynomials of Truesdel type", Publ. De, L' Inst. Math. tome, 9(23), 43-46.

96. SHRIVASTAVA, P.N. (1977) : "On a generalization of Humbert polynomials". Publ. De, L' Inst. Math. Tome, 23(36), 245-253.

97. SHRIVASTAVA, P.N. (1982) : "Generalized Rodrigue's formula and related operational relation", Pure Appl. Math. Sci., Vol. XVI, No. 1-2, 7-14.

98. SINGAL, R.P. (1968) : "Certain sums of double hypergeometric series", Rend Semi. Mat. Univ. Padova, 41, 227-234, MR 40 # 2917.

99. SINGAL, R.P. (1971) : "Transformation formula for Kampe' de Feriet hypergeometric function", Indian J. of Pure and Appl. Math., 2, 610-614.

100. SINGAL, R.P. (1972) : "Transformation formulae for the modified Kampe' de Ferit function", Math. Stud., 327-330.

101. SINGAL, R.P. (1973) : "A transformation formula for double hypergeometric series", Rocky Mount. J. Maths, 3, 377-381.

102. SINGAL, R.P.; "Some transformations of hypergeometric series", J. Soc. Math. Sci., Delhi.

103. SINGAL, R.P.; "A Watson sum for the double hypergeometric series", Ganita.

104. SLATER, L.J. (1966) : Generalized Hypergeometric Functions", Cambridge Univ. Press.

105. SNEDDON, I.N. (1966) : Mixed boundary value problems in potential theory", North-Holland, Amsterdam.

106. SONI, S.L. (1970) :"A note on the Bessel polynomials", Proc. Indian Acad. Sci. Sect. A., 71, 93-99.

107. SRIVASTAVA, G.P. (1972) : "A generalization of the theorems of Cayley and orr", 23, 81-89.

108. SRIVASTAVA, G.P. & SARAN, S. (1968) : "A theorem on Kampd de Feriet function", Proc. Camb. Phil. Soc., 64, 435-437.

109. SRIVASTAVA, G.P. & SARAN, S. (1969) : "Saalschiitzian theorems for Kampe de Feriet function", Ganita, 20, 65-73.

110. SRIVASTAVA, G.P. & SARAN, S. (1968) : "A general theorem on a Kampe de Feriet function", Sep. Math. Nat. Univ. Nac. De. Ros. Argentina, 3-4, 105-111.

111. SRIVASTAVA, H.M. (1971) : "A formal extension of certain generating functions II", Glasnik Mat. Ser. III, 6(26, 35-44.

112. SRIVASTAVA, H.M. (1972) : "Certain formulas involving Appell functions", Comment. Math. Univ. St. Paul, 21. Fasc., 1, 73-99.

113. SRIVASTAVA, H.M. (1978) : "A note on certain summation Theorems for multiple hypergeometric series", Simon Stevin, Vol. 52, 97-109.

114. SRIVASTAVA, H.M. (1979) : "Certain dual series equations involving Jacobi polynomials Nota I", Accad. Nezi. fasc., 6, 395-401.

115. SRIVASTAVA, H.M. (1980) : "Certail dual series equations involving Jacobi polynomials Nota II", Accad. Nazi. fasc., 1, 34-41.

116. SRIVASTAVA, H.M. (1984) : "Some multilinear generating functions", Univ. of Victoria Report No. DM-234-IR, 1-31, Rend. Circ. Mat. Puler (2), 33.

117. SRIVASTAVA, H.M. (1984) : "Sums of certain double q-hypergeometric series", Rend. Sem. Mat. Univ. Padova, Vol. 72, 1-8.

118. SRIVASTAVA, H.M. & DAOUST, M.C. (1971) : "Some generating functions for the Jacobi polynomials", Comment. Math. Univ. St. Paul, 20, fasc, 1, 15-21.

119. SRIVASTAVA, H.M. & MANOCHA, H.L. (1984) : "A Treatise On Generating Functions", Ellis Horwood Limited, John Wiley & Sons, New York.

120. SRIVASTAVA, H.M. & PANDA, R. (1973) : "A note on the Generalized Rice polynomials", Ranchi Univ. Math. J., 4, 65-72.

121. SRIVASTAVA, H.M. & SINGHAL, J.P. (1966) "A note on certain hypergeometric functions of two & three variables", Ganita, Vol. 17, 99-108.

122. SRIVASTAVA, H.M. & SINGHAL, J.P. (1972) : "Certain generating functions for Jacobi, Laguerre and Rice polynomials", Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom, Phys., 20, 355-363.

123. SRIVASTAVA, H.M. & SINGHAL, J.P. (1972) : "Some extensions of the Mehler formula", Proc. Amer. Math. Soc., 31, 135-141.

124. SRIVASTAVA, H.M. & SINGHAL, J.P. (1972) : "Some formulas involving the products of several Jacobi or Laguerre polynomials". Acad. Roy. Belg. Bull. Cl. Sci. (5)58, 1238-1247.

125. SRIVASTAVA, K.N. (1964) : "Some polynomials related to the Laguerre polynomials", J. Indian Math. Soc., 28, 43-50.

126. SRIVASTAVA,R.P. (1964) : "Dual series relations. IV: Dual relations involving series of Jacobi polynomials". Proc. Roy. Soc. Edinburgh Sect. A., 66, 185-191.

127. THAKARE, N.K. (1974) : "Some dual series equations involving Jacobi polynomials". Z. Angew. Math. Mech., 54, 283-284.

128. THAKARE, N.K. (1980) : "Multilinear generating functions for some orthogonal polynomials." Proc. Nat. Acad. Sci. India Sect. A, 50, 1-6.

129. THAKARE, N.K. (1980) : "Note on some formulas involving the products of several Jacobi or Laguerre polynomials". Indian J.Pure Appl. Math., 11, 1158-1161.

130. THAKARE, N.K. & KARANDE, B.K. (1973) : "Some properties of orthogonal polynomials related to Hermite polynomials". Bull. Math. Soc. Sci. Math. R.S. Roumannie (N.S.), 17, 57-69.

131. TOSCANO, L. (1949) : "I polinomi ipergeometrici nel calcolo delle differenze finite". Boll. Un. Mat. Ital., 4, 398-409.

132. TOSCANO, L. (1955) : "Differenze finite e derivate dei polinomi di Jcobi" Matematiche (Catania), 10, 44-56.

133. TOSCANO, L. (1956) : "Una gneralizzazione dei polinomi di Laguerre" di Giron. Mat. Battaglini, 84, 123-138.

134. TOSCANO, L.; (1960) : "Polinomi di Jacobi e piu generali nel Calcolo delle differenze finite". Mathematiche (Catania), 15, 41-53.

135. TOSCANO, L. (1968) : "Operatori differenziali e polinomi di Laguerre" Mathematiche (Catania), 23, 197-223.

136. TOSCANO, L. (1970) : "Generalizzazioni dei polinomi di Laguerre e dei polinomi attuariali", Riv. Mat. Univ. Parma, 11, 191-226.

137. TRICOMI, F.G. (1951) : "A class of non-orthogonal polynomials related to those of Laguerre". Journal d' analyse Mathematique, 1, 209-231.

138. TRUESDELL, C. (1948) : "An essay toward a unified theory of special functions". Princetown : University Press.

139. WEISNER, L. (1955) : "Group theoretic origin of certain generating functions". Pacific J. Math. 4, Supp. 2, 1033-1039.

140. ZEITLIN, D. (1967) : "On generating functions and a formula of Chaudhri". Amer. Math. Monthly, 74, 1056-1062.

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